

Periods of Drinfeld modules and local shtukas with complex multiplication

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Abstract

Colmez [Col93] conjectured a product formula for periods of abelian varieties over number fields with complex multiplication and proved it in some cases. His conjecture is equivalent to a formula for the Faltings height of CM abelian varieties in terms of the logarithmic derivatives at $s = 0$ of certain Artin L -functions.

In a series of articles we investigate the analog of Colmez's theory in the arithmetic of function fields. There abelian varieties are replaced by Drinfeld modules and their higher dimensional generalizations, so-called A -motives. In the present article we prove the product formula for the Carlitz module and we compute the valuations of the periods of a CM A -motive at all finite places in terms of Artin L -series. The latter is achieved by investigating the local shtukas associated with the A -motive. *Mathematics Subject Classification (2000)*: 11G09, (11R42, 11R58, 14L05)

1 Introduction

In [Col93] P. Colmez considers product formulas for periods of abelian varieties. Let X be an abelian variety defined over a number field K with complex multiplication by the ring of integers in a CM-field E and of CM-type Φ . Let \mathbb{Q}^{alg} be the algebraic closure of \mathbb{Q} in \mathbb{C} , let $H_E := \text{Hom}_{\mathbb{Q}}(E, \mathbb{Q}^{\text{alg}})$ be the set of all ring homomorphisms $E \hookrightarrow \mathbb{Q}^{\text{alg}}$ and assume that K contains $\psi(E)$ for every $\psi \in H_E$. For a $\psi \in H_E$ let $\omega_{\psi} \in H_{\text{dR}}^1(X, K)$ be a non-zero cohomology class such that $a^* \omega_{\psi} = \psi(a) \cdot \omega_{\psi}$ for all $a \in E$. For every embedding $\eta: K \hookrightarrow \mathbb{Q}^{\text{alg}}$, let X^{η} and ω_{ψ}^{η} be deduced from X and ω_{ψ} by base extension. Let $(u_{\eta})_{\eta} \in \prod_{\eta \in H_K} H_1(X^{\eta}(\mathbb{C}), \mathbb{Z})$ be a family of cycles compatible with complex conjugation. Let v be a place of \mathbb{Q} . If $v = \infty$ the de Rham isomorphism between Betti and de Rham cohomology yields a complex number $\int_{u_{\eta}} \omega_{\psi}^{\eta}$ and its absolute value $|\int_{u_{\eta}} \omega_{\psi}^{\eta}|_{\infty} \in \mathbb{R}$. If v corresponds to a prime number $p \in \mathbb{Z}$, we fix an inclusion $\mathbb{Q}^{\text{alg}} \hookrightarrow \mathbb{Q}_p^{\text{alg}}$. With this data Colmez [Col93] associates a period $\int_{u_{\eta}} \omega_{\psi}^{\eta}$ in Fontaine's p -adic period field \mathbf{B}_{dR} and an absolute value $|\int_{u_{\eta}} \omega_{\psi}^{\eta}|_v \in \mathbb{R}$. He considers the product $\prod_v \prod_{\eta \in H_K} |\int_{u_{\eta}} \omega_{\psi}^{\eta}|_v$ and (after some modifications) conjectures that this product evaluates to 1; see [Col93, Conjecture 0.1] for the precise formulation. This conjecture is equivalent to a conjectural formula for the Faltings height of a CM abelian variety in terms of the logarithmic derivatives at $s = 0$ of certain Artin L -functions. Colmez proves the conjectures when E is an abelian extension of \mathbb{Q} . On the way, he computes $\prod_{\eta \in H_K} |\int_{u_{\eta}} \omega_{\psi}^{\eta}|_v$ at a finite place v in terms of the local factor at v of the Artin L -series associated with an Artin character $a_{E, \psi, \Phi}^0: \text{Gal}(\mathbb{Q}^{\text{alg}}/\mathbb{Q}) \rightarrow \mathbb{C}$ that only depends on E , ψ and Φ but not on X and v ; see [Col93, Théorème I.3.15]. There has been further progress on Colmez's conjecture by Obus [Obu13], Yang [Yan13], Andreatta, Goren, Howard, Madapusi Pera [AGHM15], Yuan, Zhang [YZ15] and others.

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Our goal in this article is to develop the analog of Colmez's theory in the "Arithmetic of function fields". Here abelian varieties are replaced by Drinfeld modules [Dri76, Gos96] and their higher dimensional generalizations, so-called A -motives. To define them let \mathbb{F}_q be a finite field with q elements, let C be a smooth projective, geometrically irreducible curve over \mathbb{F}_q , let $\infty \in C$ be a fixed closed point and let $A := \Gamma(C \setminus \{\infty\}, \mathcal{O}_C)$ be the ring of regular functions on C outside ∞ . Let Q be the fraction field of A and let K be a finite field extension of Q contained in a fixed algebraic closure Q^{alg} of Q . We write $A_K := A \otimes_{\mathbb{F}_q} K$ and consider the endomorphism $\sigma := \text{id}_A \otimes \text{Frob}_{q,K}$ of A_K , where $\text{Frob}_{q,K}(b) = b^q$ for $b \in K$. For an A_K -module M we set $\sigma^* M := M \otimes_{A_K, \sigma} A_K$ and for a homomorphism $f: M \rightarrow N$ of A_K -modules we set $\sigma^* f := f \otimes \text{id}_{A_K}: \sigma^* M \rightarrow \sigma^* N$. Let $\gamma: A \rightarrow K$ be the inclusion $A \subset Q \subset K$, and set $\mathcal{J} := (a \otimes 1 - 1 \otimes \gamma(a): a \in A) \subset A_K$. Then γ can be recovered as the homomorphism $A \rightarrow A_K/\mathcal{J} = K$.

Definition 1.1. An A -motive of rank r over K is a pair $\underline{M} = (M, \tau_M)$ consisting of a locally free A_K -module M of rank r and an isomorphism $\tau_M: \sigma^* M|_{\text{Spec } A_K \setminus V(\mathcal{J})} \xrightarrow{\sim} M|_{\text{Spec } A_K \setminus V(\mathcal{J})}$ of the associated sheaves outside $V(\mathcal{J}) \subset \text{Spec } A_K$. We write $\text{rk } \underline{M} := r$. A *morphism* between A -motives $f: (M, \tau_M) \rightarrow (N, \tau_N)$ is an A_K -homomorphism $f: M \rightarrow N$ with $f \circ \tau_M = \tau_N \circ \sigma^* f$.

A -motives generalize Anderson's t -motives [And86] and Drinfeld modules. An A -motive has various cohomology realizations, for example the *de Rham realization* $H_{\text{dR}}^1(\underline{M}, K) := \sigma^* M/\mathcal{J} \cdot \sigma^* M$ and for each maximal ideal $v \subset A$ a *v -adic étale realization* $H_v^1(\underline{M}, A_v)$ where A_v is the v -adic completion of A ; see Definition 2.3 below. We let Q_v be the fraction field of A_v , and we let Q_∞ be the ∞ -adic completion of Q and \mathbb{C}_∞ be the completion of a fixed algebraic closure of Q_∞ . We fix a Q -embedding $Q^{\text{alg}} \hookrightarrow \mathbb{C}_\infty$ and consider the base extension of \underline{M} to \mathbb{C}_∞ . There is a notion of \underline{M} being *uniformizable* and a uniformizable \underline{M} has a *Betti realization* $H_{\text{Betti}}^1(\underline{M}, A)$. These realizations are related by period isomorphisms

$$\begin{aligned} h_{\text{Betti}, v}: H_{\text{Betti}}^1(\underline{M}, A) \otimes_A A_v &\xrightarrow{\sim} H_v^1(\underline{M}, A_v) \quad \text{and} \\ h_{\text{Betti}, \text{dR}}: H_{\text{Betti}}^1(\underline{M}, A) \otimes_A \mathbb{C}_\infty &\xrightarrow{\sim} H_{\text{dR}}^1(\underline{M}, K) \otimes_K \mathbb{C}_\infty. \end{aligned}$$

Also for a *place* v of Q , that is a closed point $v \subset C$, let \mathbb{F}_v be its residue field and set $q_v := \#\mathbb{F}_v = q^{[\mathbb{F}_v: \mathbb{F}_q]}$. Let $z := z_v \in Q$ be a uniformizing parameter at v . Then there is a canonical isomorphism $A_v = \mathbb{F}_v[[z_v]]$. Let $\zeta := \zeta_v := \gamma(z_v)$ denote the image of z_v in K . We simply write z , resp. ζ for the elements $z \otimes 1$, resp. $1 \otimes \zeta$ of $Q \otimes_{\mathbb{F}_q} K$. Then the power series ring $K[[z - \zeta]]$ in the "variable" $z - \zeta$ is canonically isomorphic to the completion of the local ring of $C_K := C \times_{\mathbb{F}_q} K$ at $V(\mathcal{J})$, and thus independent of v . We always consider the embedding $Q \hookrightarrow K[[z - \zeta]]$ given by $z \mapsto z = \zeta + (z - \zeta)$. The de Rham realization lifts to $H_{\text{dR}}^1(\underline{M}, K[[z - \zeta]]) := \sigma^* M \otimes_{A_K} K[[z - \zeta]]$, and the vector space $H_{\text{dR}}^1(\underline{M}, K[[z - \zeta]])[\frac{1}{z - \zeta}]$ over the field $K((z - \zeta)) := K[[z - \zeta]][\frac{1}{z - \zeta}]$ contains the $K[[z - \zeta]]$ -lattice $\mathbf{q}^M := \tau_M^{-1}(M \otimes_{A_K} K[[z - \zeta]])$, which is called the *Hodge-Pink lattice* of \underline{M} and is the analog of the Hodge-filtration of an abelian variety; see [HK15, Remark 5.13].

If $v \neq \infty$ we also fix a Q -embedding of Q^{alg} into a fixed algebraic closure Q_v^{alg} of Q_v and we let \mathbb{C}_v be the v -adic completion of Q_v^{alg} . Again we denote the image of z_v in Q_v^{alg} and \mathbb{C}_v by ζ_v . We let $K_v \subset Q_v^{\text{alg}}$ be the induced completion of K and we let R be its valuation ring. If \underline{M} has good reduction over R there is a period isomorphism

$$h_{v, \text{dR}}: H_v^1(\underline{M}, A_v) \otimes_{A_v} \mathbb{C}_v((z_v - \zeta_v)) \xrightarrow{\sim} H_{\text{dR}}^1(\underline{M}, K[[z_v - \zeta_v]]) \otimes_{K[[z_v - \zeta_v]]} \mathbb{C}_v((z_v - \zeta_v)).$$

The field $\mathbb{C}_v((z_v - \zeta_v))$ is the analog of Fontaine's p -adic period field \mathbf{B}_{dR} ; see [HK15, Remark 4.3].

Now we say that \underline{M} has *complex multiplication* if $Q\text{End}_K(\underline{M}) := \text{End}_K(\underline{M}) \otimes_A Q$ contains a commutative, semi-simple Q -algebra E of dimension $\dim_Q E = \text{rk } \underline{M}$. Here semi-simple means that E is a product of fields. Note that we do not assume that E is itself a field. Let \mathcal{O}_E be the integral closure of A in E . It is a locally free A -module of $\text{rk}_A \mathcal{O}_E = \dim_Q E$. We let $H_E := \text{Hom}_Q(E, Q^{\text{alg}})$ be the set of Q -homomorphisms $\psi: E \rightarrow Q^{\text{alg}}$ and we assume that K contains $\psi(E)$ for every $\psi \in H_E$. Then by

Lemma A.3 in the appendix there is a decomposition $E \otimes_Q K[[z - \zeta]] = \prod_{\psi \in H_E} K[[y_\psi - \psi(y_\psi)]]$, where y_ψ is a uniformizer at a place of E such that $\psi(y_\psi) \neq 0$. Correspondingly $H_{\text{dR}}^1(\underline{M}, K[[z - \zeta]])$ decomposes into eigenspaces

$$H^\psi(\underline{M}, K[[y_\psi - \psi(y_\psi)]]) := H_{\text{dR}}^1(\underline{M}, K[[z - \zeta]]) \otimes_{E \otimes_Q K[[z - \zeta]]} K[[y_\psi - \psi(y_\psi)]]$$

each of which is free of rank 1 over $K[[y_\psi - \psi(y_\psi)]]$. There are integers d_ψ such that the Hodge-Pink lattice is $\mathfrak{q}^{\underline{M}} = \prod_{\psi} (y_\psi - \psi(y_\psi))^{-d_\psi} H^\psi(\underline{M}, K[[y_\psi - \psi(y_\psi)]])$. The tuple $\Phi := (d_\psi)_{\psi \in H_E}$ is the *CM-type* of \underline{M} .

If we fix elements $u \in H_{1, \text{Betti}}(\underline{M}, Q) := \text{Hom}_A(H_{\text{Betti}}^1(\underline{M}, A), Q)$ and $\omega \in H_{\text{dR}}^1(\underline{M}, K[[z - \zeta]])$ we can define

$$\langle \omega, u \rangle_\infty := u \otimes \text{id}_{\mathbb{C}_\infty} (h_{\text{Betti}, \text{dR}}^{-1}(\omega \bmod z - \zeta)) \in \mathbb{C}_\infty \quad \text{and} \quad (1.1)$$

$$|\int_u \omega|_\infty := |\langle \omega, u \rangle_\infty|_\infty \in \mathbb{R}, \quad (1.2)$$

where $|\cdot|_v$ is the normalized absolute value on \mathbb{C}_v with $|\zeta_v|_v = (\#\mathbb{F}_v)^{-1} = q_v^{-1}$ for every place v . We also consider the valuation v on \mathbb{C}_v with $v(\zeta_v) = 1$. The expressions in (1.1) and (1.2) only depend on the image of ω in $H_{\text{dR}}^1(\underline{M}, K)$. Also at a finite place $v \neq \infty$ of Q we consider on elements $x \neq 0$ of the discretely valued field $\mathbb{C}_v((z_v - \zeta_v))$ the valuation $\hat{v}(x) := \text{ord}_{z_v - \zeta_v}(x)$, and in addition we define $|x|_v := |((z_v - \zeta_v)^{-\hat{v}(x)} \cdot x) \bmod z_v - \zeta_v|_v$ and $v(x) := -\log |x|_v / \log q_v$ induced from $((z_v - \zeta_v)^{-\hat{v}(x)} \cdot x) \bmod z_v - \zeta_v \in \mathbb{C}_v$. Note that $|x|_v$ and $v(x)$ are not a norm, respectively a valuation, because they do not satisfy the triangle inequality. The value $|x|_v$ does not depend on the choice of the uniformizer z_v of A_v , because if $\tilde{z}_v = \sum_{n=0}^\infty b_n z_v^n =: f(z_v)$ with $b_n \in \mathbb{F}_v$ is another uniformizer and $\tilde{\zeta}_v = f(\zeta_v)$, then $\frac{\tilde{z}_v - \tilde{\zeta}_v}{z_v - \zeta_v} \equiv f'(\zeta_v) \bmod z_v - \zeta_v$ in $\mathcal{O}_{\mathbb{C}_v}[[z_v]] = \mathbb{F}_v[[z_v]] \hat{\otimes}_{\mathbb{F}_v, \gamma} \mathcal{O}_{\mathbb{C}_v}$ by Lemma A.1 in the appendix and $f'(\zeta_v) \in \mathbb{F}_v[[\zeta_v]]^\times$ with inverse $\frac{dz_v}{d\tilde{z}_v}|_{\tilde{z}_v = \tilde{\zeta}_v}$. We define

$$\langle \omega, u \rangle_v := u \otimes_{\mathbb{C}_v((z_v - \zeta_v))} (h_{\text{Betti}, v}^{-1} \circ h_{v, \text{dR}}^{-1}(\omega)) \in \mathbb{C}_v((z_v - \zeta_v)) \quad \text{and} \quad (1.3)$$

$$|\int_u \omega|_v := |\langle \omega, u \rangle_v|_v := |((z_v - \zeta_v)^{-\hat{v}(\langle \omega, u \rangle_v)} \cdot \langle \omega, u \rangle_v) \bmod z_v - \zeta_v|_v \in \mathbb{R}. \quad (1.4)$$

We will show in Theorem 4.24 below that if E is separable over Q and if $\omega \in H^\psi(\underline{M}, K[[y_\psi - \psi(y_\psi)]])$ has non-zero image in $H_{\text{dR}}^1(\underline{M}, K)$, then the absolute value (1.4) only depends on that image.

With these definitions we can now consider the product $\prod_v |\int_u \omega|_v$, or equivalently its logarithm $\log \prod_v |\int_u \omega|_v = -\sum_v v(\int_u \omega) \log q_v$. Like in Colmez's theory, these products or sums do not converge and one has to give a convergent interpretation to their finite parts $\prod_{v \neq \infty} |\int_u \omega|_v$, respectively $-\sum_{v \neq \infty} v(\int_u \omega) \log q_v$; see Convention 1.4 below. To formulate the convention we make the following

Definition 1.2. For $F = Q$ or $F = Q_v$ let F^{sep} be the separable closure of F in F^{alg} and let $\mathcal{G}_F := \text{Gal}(F^{\text{sep}}/F)$. For a finite field extension F' of F let $H_{F'} := \text{Hom}_F(F', F^{\text{alg}})$ be the set of F -homomorphisms $\psi: F' \rightarrow F^{\text{alg}}$. Let $\mathcal{C}(\mathcal{G}_F, \mathbb{Q})$ be the \mathbb{Q} -vector space of locally constant functions $a: \mathcal{G}_F \rightarrow \mathbb{Q}$ and let $\mathcal{C}^0(\mathcal{G}_F, \mathbb{Q})$ be the subspace of those functions which are constant on conjugacy classes, that is, which satisfy $a(h^{-1}gh) = a(g)$ for all $g, h \in \mathcal{G}_F$. Then the \mathbb{C} -vector space $\mathcal{C}^0(\mathcal{G}_F, \mathbb{Q}) \otimes_{\mathbb{Q}} \mathbb{C}$ is spanned by the traces of representations $\rho: \mathcal{G}_F \rightarrow \text{GL}_n(\mathbb{C})$ with open kernel for varying n by [Ser77, § 2.5, Theorem 6]. Via the fixed embedding $Q^{\text{sep}} \hookrightarrow Q_v^{\text{sep}}$ we consider the induced inclusion $\mathcal{G}_{Q_v} \subset \mathcal{G}_Q$ and morphism $\mathcal{C}(\mathcal{G}_Q, \mathbb{Q}) \rightarrow \mathcal{C}(\mathcal{G}_{Q_v}, \mathbb{Q})$. If χ is the trace of a representation $\rho: \mathcal{G}_Q \rightarrow \text{GL}_n(\mathbb{C})$ with open kernel we let $L(\chi, s) := \prod_{\text{all } v} L_v(\chi, s)$, respectively $L^\infty(\chi, s) := \prod_{v \neq \infty} L_v(\chi, s)$ be the Artin L -function of ρ , respectively without the factor at ∞ . It only depends on χ and converges for all $s \in \mathbb{C}$ with

$\operatorname{Re}(s) > 1$; see [Ros02, pp. 126ff]. We also set

$$Z(\chi, s) := \frac{\frac{d}{ds}L(\chi, s)}{L(\chi, s)} = - \sum_{\text{all } v} Z_v(\chi, s) \log q_v \quad \text{and} \quad (1.5)$$

$$Z^\infty(\chi, s) := \frac{\frac{d}{ds}L^\infty(\chi, s)}{L^\infty(\chi, s)} = - \sum_{v \neq \infty} Z_v(\chi, s) \log q_v \quad \text{with} \quad (1.6)$$

$$Z_v(\chi, s) := \frac{\frac{d}{ds}L_v(\chi, s)}{-L_v(\chi, s) \cdot \log q_v} = \frac{\frac{d}{dq_v^{-s}}L_v(\chi, s)}{q_v^s \cdot L_v(\chi, s)}. \quad (1.7)$$

Moreover, we let \mathfrak{f}_χ be the Artin conductor of χ . It is an effective divisor $\mathfrak{f}_\chi = \sum_v \mu_{\text{Art},v}(\chi) \cdot (v)$ on C ; see [Ser79, Chapter VI, §§ 2,3], where $\mu_{\text{Art},v}(\chi)$ is denoted $f(\chi, v)$. We set

$$\mu_{\text{Art}}(\chi) := \deg(\mathfrak{f}_\chi) \log q := \sum_{\text{all } v} \mu_{\text{Art},v}(\chi) [\mathbb{F}_v : \mathbb{F}_q] \log q = \sum_{\text{all } v} \mu_{\text{Art},v}(\chi) \log q_v \quad \text{and} \quad (1.8)$$

$$\mu_{\text{Art}}^\infty(\chi) := \sum_{v \neq \infty} \mu_{\text{Art},v}(\chi) \log q_v. \quad (1.9)$$

In particular, only finitely many values $\mu_{\text{Art},v}(\chi)$ are non-zero. By linearity we extend $Z^\infty(\cdot, s)$ and μ_{Art}^∞ to all $a \in \mathcal{C}^0(\mathcal{G}_Q, \mathbb{Q})$ and $Z_v(\cdot, s)$ and $\mu_{\text{Art},v}$ to all $a \in \mathcal{C}^0(\mathcal{G}_{Q_v}, \mathbb{Q})$. The map $Z_v(\cdot, s)$ takes values in $\mathbb{Q}(q_v^{-s})$.

In terms of this definition we prove in this article a formula for $|\int_u \omega|_v$ for a uniformizable A -motive \underline{M} over K with complex multiplication by a semi-simple *separable* CM-algebra E of CM-type $\Phi = (d_\varphi)_{\varphi \in H_E}$ as follows. By results of Schindler [Sch09] there is an A -motive \underline{M}' isogenous to \underline{M} such that the integral closure \mathcal{O}_E of A in E is contained in $\operatorname{End}_K(\underline{M}')$ and \underline{M} and \underline{M}' have good reduction everywhere after replacing K by a finite separable extension. Moreover, every A -motive over a field extension of Q with $\mathcal{O}_E \subset \operatorname{End}_K(\underline{M})$ is already defined over a *finite separable* extension K of Q . So we assume that $\mathcal{O}_E \subset \operatorname{End}_K(\underline{M})$, that K is Galois over Q and contains $\psi(E)$ for all $\psi \in H_E$, and that \underline{M} has good reduction at all primes of K . For $\psi \in H_E$ we define the functions

$$a_{E,\psi,\Phi}: \mathcal{G}_Q \rightarrow \mathbb{Z}, \quad g \mapsto d_{g\psi} \quad \text{and} \quad (1.10)$$

$$a_{E,\psi,\Phi}^0: \mathcal{G}_Q \rightarrow \mathbb{Q}, \quad g \mapsto \frac{1}{\#H_K} \sum_{\eta \in H_K} d_{\eta^{-1}g\eta\psi} \quad (1.11)$$

which factor through $\operatorname{Gal}(K/Q)$. In particular, $a_{E,\psi,\Phi} \in \mathcal{C}(\mathcal{G}_Q, \mathbb{Q})$ and $a_{E,\psi,\Phi}^0 \in \mathcal{C}^0(\mathcal{G}_Q, \mathbb{Q})$ is independent of K .

Note that $H_{1,\text{Betti}}(\underline{M}, Q)$ is a free E -module of rank 1 by [BH09, Lemma 7.2] and that the eigenspace $H^\psi(\underline{M}, K) := H^\psi(\underline{M}, K[[y_\psi - \psi(y_\psi)]) / (y_\psi - \psi(y_\psi)) H^\psi(\underline{M}, K[[y_\psi - \psi(y_\psi)]])$ in $H_{\text{dR}}^1(\underline{M}, K)$ of the character $\psi: E \rightarrow K$ is a K -vector space of dimension 1 by Proposition 3.8 below. For an E -generator $u \in H_{1,\text{Betti}}(\underline{M}, Q)$ and a generator $\omega_\psi \in H^\psi(\underline{M}, K[[y_\psi - \psi(y_\psi)]])$ as $K[[y_\psi - \psi(y_\psi)]]$ -module we define integers $v(\omega_\psi)$ and $v_\psi(u)$ for all $v \neq \infty$ which are all zero except for finitely many. Let $\mathcal{O}_{E_v} := \mathcal{O}_E \otimes_A A_v$ and let $c \in E_v := E \otimes_Q Q_v$ be such that $c^{-1}u$ is an \mathcal{O}_{E_v} -generator of $H_{1,\text{Betti}}(\underline{M}, A) \otimes_A A_v$, which exists because \mathcal{O}_{E_v} is a product of discrete valuation rings. Then c is unique up to multiplication by an element of $\mathcal{O}_{E_v}^\times$ and we set

$$v_\psi(u) := v(\psi(c)) \in \mathbb{Z}, \quad (1.12)$$

where we extend $\psi \in H_E$ by continuity to $\psi: E_v \rightarrow Q_v^{\text{alg}}$. Also let $\underline{M} = (\mathcal{M}, \tau_{\mathcal{M}})$ be an A -motive over $R := \mathcal{O}_{K_v}$ with good reduction and $\underline{M} \otimes_{\mathcal{O}_{K_v}} K_v \cong \underline{M} \otimes_K K_v$; see Example 2.2. Then there is an element

$x \in K_v^\times$, unique up to multiplication by R^\times , such that $x^{-1}\omega_\psi \bmod y_\psi - \psi(y_\psi)$ is an R -generator of the free R -module of rank one

$$H^\psi(\underline{M}, R) := \{ \omega \in H_{\text{dR}}^1(\underline{M}, R) := \sigma^* \mathcal{M} \otimes_{A_R, \gamma \otimes \text{id}_R} R : [b]^* \omega = \psi(b) \cdot \omega \ \forall b \in \mathcal{O}_E \},$$

and we set

$$v(\omega_\psi) := v(x) \in \mathbb{Z}. \quad (1.13)$$

This value only depends on the image of ω_ψ in $H_{\text{dR}}^1(\underline{M}, K)$. It also does not depend on the choice of the model \underline{M} with good reduction, because all such models are isomorphic over R by [Gar03, Proposition 2.13(ii)]. In this situation our main result is the following

Theorem 1.3. *Let ω_ψ be a generator of the $K[[y_\psi - \psi(y_\psi)]]$ -module $H^\psi(\underline{M}, K[[y_\psi - \psi(y_\psi)]])$. For every $\eta \in H_K$ let \underline{M}^η and $\omega_\psi^\eta \in H^{\eta\psi}(\underline{M}^\eta, K[[y_{\eta\psi} - \eta\psi(y_{\eta\psi})]])$ be obtained by extension of scalars via η , and choose an E -generator $u_\eta \in H_{1, \text{Betti}}(\underline{M}^\eta, Q)$. Then for every place $v \neq \infty$ of C we have*

$$\frac{1}{\#H_K} \sum_{\eta \in H_K} v(\int_{u_\eta} \omega_\psi^\eta) = Z_v(a_{E, \psi, \Phi}^0, 1) - \mu_{\text{Art}, v}(a_{E, \psi, \Phi}^0) + \frac{1}{\#H_K} \sum_{\eta \in H_K} (v(\omega_\psi^\eta) + v_{\eta\psi}(u_\eta)).$$

We will prove this theorem at the end of Section 4 by using the *local shtuka* at v attached to \underline{M} . The latter is an analog of the Dieudonné-module of the p -divisible group attach to an abelian variety; see [Har09, § 3.2]. The theorem allows us to make the following convention which is the analog of [Col93, Convention 0].

Convention 1.4. Let $\Sigma = \sum_{v \neq \infty} x_v$ be a series. We suppose that there exists an element $a \in \mathcal{C}^0(\mathcal{G}_Q, \mathbb{Q})$ such that $x_v = -Z_v(a, 1) \log q_v$ for all v except for finitely many. Then we let $a^* \in \mathcal{C}^0(\mathcal{G}_Q, \mathbb{Q})$ be defined by $a^*(g) := a(g^{-1})$. We further assume that $Z^\infty(a^*, s)$ does not have a pole at $s = 0$, and we give Σ the value

$$-Z^\infty(a^*, 0) - \mu_{\text{Art}}^\infty(a) - 2 \cdot \text{genus}(C) \cdot a(1) \log q + \sum_{v \neq \infty} (x_v + Z_v(a, 1) \log q_v) \quad (1.14)$$

inspired by Weil's [Wei48, p. 82] functional equation

$$Z(\chi, 1-s) = -Z(\chi^*, s) - (2 \cdot \text{genus}(C) - 2)\chi(1) \log q - \mu_{\text{Art}}(\chi)$$

deprived of the summands at ∞ .

The Convention 1.4 and the Theorem 1.3 allow us to give a convergent interpretation to the sum $-\sum_v \sum_{\eta \in H_K} v(\int_{u_\eta} \omega_\psi^\eta) \log q_v$ and the product $\prod_v \prod_{\eta \in H_K} |\int_{u_\eta} \omega_\psi^\eta|_v$, and we can ask whether this product is 1. We will answer this question in a sequel to this article, where we also discuss its consequences for the Faltings height of CM A -motives similar to [Col93, Théorème 0.3 and Conjecture 0.4], and conditions under which $Z^\infty(a^*, s)$ does not have a pole at $s = 0$. Nevertheless, let us explain here the easiest case of the *Carlitz motive* which is related to the zeta function of $\mathbb{F}_q[t]$ and is the analog of the multiplicative group $\mathbb{G}_{m, \mathbb{Q}}$ considered by Colmez.

Example 1.5. Let $A = \mathbb{F}_q[t]$ and $C = \mathbb{P}_{\mathbb{F}_q}^1$. Let $K = \mathbb{F}_q(\vartheta)$ be the rational function field in the variable ϑ and let $\gamma : A \rightarrow K$ be given by $\gamma(t) = \vartheta$. We also set $z := z_\infty := \frac{1}{t}$ and $\zeta := \zeta_\infty := \frac{1}{\vartheta}$. It satisfies $|\zeta|_\infty = q^{-1} < 1$. The *Carlitz motive* over K is the A -motive $\underline{\mathcal{C}} = (K[t], \tau_{\mathcal{C}} = z - \zeta)$ which is associated with the Carlitz module; see [Car35] or [Gos96, Chapter 3]. It has rank 1 and dimension 1, and complex multiplication by the ring of integers A in $E := Q$ with CM-type $\Phi = (d_{\text{id}})$, where $H_E = \{\text{id}\}$ and $d_{\text{id}} = 1$. As is well known, its cohomology satisfies $H_{\text{dR}}^1(\underline{\mathcal{C}}, K[[z - \zeta]]) = K[[z - \zeta]]$ and $H_{\text{Betti}}^1(\underline{\mathcal{C}}, A) = A \cdot \beta \ell_\zeta^-$, where $\beta \in \mathbb{C}_\infty$ satisfies $\beta^{q-1} = -\zeta$ and $\ell_\zeta^- := \prod_{i=0}^\infty (1 - \zeta^{q^i} t)$; see for example [HJ16, Example 3.32]. We denote the generator 1 of $H_{\text{dR}}^1(\underline{\mathcal{C}}, K[[z - \zeta]])$ by ω and we take $u \in H_{1, \text{Betti}}(\underline{\mathcal{C}}, \mathbb{F}_q[t])$ as the generator which is dual to $\beta \ell_\zeta^- \in H_{\text{Betti}}^1(\underline{\mathcal{C}}, \mathbb{F}_q[t])$. The de Rham isomorphism $h_{\text{Betti}, \text{dR}}$ sends $\beta \ell_\zeta^-$ to

$$\sigma^*(\beta \ell_\zeta^-) \cdot \omega = \beta^q \sigma^*(\ell_\zeta^-) \cdot \omega \in H_{\text{dR}}^1(\underline{M}, \mathbb{C}[[z - \zeta]]) = \mathbb{C}[[z - \zeta]] \cdot \omega,$$

respectively to $\beta^q \sigma^*(\ell_\zeta^-)|_{t=\vartheta} \cdot \omega = \beta^q \prod_{i=1}^\infty (1 - \zeta^{q^i-1}) \cdot \omega \in H_{\text{dR}}^1(\underline{M}, \mathbb{C}) = \mathbb{C} \cdot \omega$. Here the coefficient $\beta^q \prod_{i=1}^\infty (1 - \zeta^{q^i-1})$ is the function field analog of the complex number $(2i\pi)^{-1}$, the inverse of the period of the multiplicative group $\mathbb{G}_{m, \mathbb{Q}}$. We obtain

$$|\int_u \omega|_\infty = |(\beta^q \prod_{i=1}^\infty (1 - \zeta^{q^i-1}))^{-1}|_\infty = |\beta|_\infty^{-q} = q^{q/(q-1)}.$$

At a finite place $v \subset \mathbb{F}_q[t]$ let $v = (z_v)$ and $\zeta_v = \gamma(z_v)$. Then $H_v^1(\underline{M}, A_v) = A_v \cdot (\ell_{\zeta_v}^+)^{-1}$, where $\ell_{\zeta_v}^+ := \sum_{n=0}^\infty \ell_n z_v^n \in \mathbb{C}_v[[z_v]]$ with $\ell_0^{q_v-1} = -\zeta_v$ and $\ell_n^{q_v} + \zeta_v \ell_n = \ell_{n-1}$; see [HK15, Example 4.16]. This implies $|\ell_n| = |\zeta_v|^{q_v^{-n}/(q_v-1)} < 1$. The v -adic comparison isomorphism $h_{v, \text{dR}}$ sends the generator $\ell_{\zeta_v}^+$ of $H_v^1(\underline{M}, A_v)$ to

$$(z_v - \zeta_v)^{-1} (\ell_{\zeta_v}^+)^{-1} \cdot \omega \in H_{\text{dR}}^1(\underline{M}, \mathbb{C}_v((z_v - \zeta_v))) = \mathbb{C}_v((z_v - \zeta_v)) \cdot \omega,$$

where the coefficient of ω is the v -adic Carlitz period which has a pole of order one at $z_v = \zeta_v$. So $\langle \omega, u \rangle_v = (z_v - \zeta_v) \ell_{\zeta_v}^+$ has $\hat{v}(\langle \omega, u \rangle_v) = 1$ and

$$|\int_u \omega|_v = |\ell_{\zeta_v}^+ \bmod z_v - \zeta_v|_v = \left| \sum_{n=0}^\infty \ell_n \zeta_v^n \right|_v = |\ell_0|_v = q_v^{-1/(q_v-1)}$$

So the product $\prod_v |\int_u \omega|_v$ of the norms at all places has logarithm

$$\log \prod_{\text{all } v} |\int_u \omega|_v = \log |\int_u \omega|_\infty + \log \prod_{v \neq \infty} |\int_u \omega|_v = \frac{q}{q-1} \log q + \sum_{v \neq \infty} \frac{-1}{q_v-1} \log q_v. \quad (1.15)$$

Note that this series is not convergent, but that the sum over $v \neq \infty$ is equal to $\frac{\zeta'_A(1)}{\zeta_A(1)}$ and the summand at ∞ is equal to $\frac{\zeta'_A(0)}{\zeta_A(0)}$, where ζ_A is the zeta function associated with A . Namely, the zeta functions are defined as the following products which converge for $s \in \mathbb{C}$ with $\text{Re}(s) > 1$

$$\begin{aligned} \zeta_C(s) &:= \prod_{\text{all } v} (1 - (\#\mathbb{F}_v)^{-s})^{-1} = \prod_{\text{all } v} (1 - q_v^{-s})^{-1} = \frac{1}{(1 - q^{-s})(1 - q^{1-s})} \quad \text{and} \\ \zeta_A(s) &:= \prod_{v \neq \infty} (1 - (\#\mathbb{F}_v)^{-s})^{-1} = \prod_{v \neq \infty} (1 - q_v^{-s})^{-1} = \frac{1}{1 - q^{1-s}}. \end{aligned}$$

In particular $(1 - q_v^{-s})^{-1} = L_v(\mathbb{1}, 1)$ and $\zeta_A(s) = L^\infty(\mathbb{1}, s)$ where $\mathbb{1}: g \mapsto 1$ is the trivial character in $\mathcal{C}^0(\mathcal{G}_Q, \mathbb{Q})$, which equals $a_{Q, \text{id}, \Phi} = a_{Q, \text{id}, \Phi}^0$, and $\frac{\zeta'_A(s)}{\zeta_A(s)} = Z^\infty(\mathbb{1}, s)$. Applying Convention 1.4 with $a = \mathbb{1}$ and $\mu_{\text{Art}}(\mathbb{1}) = 0$ and $\text{genus}(\mathbb{P}_{\mathbb{F}_q}^1) = 0$ we obtain

$$\begin{aligned} \frac{q}{q-1} \log q + \sum_{v \neq \infty} \frac{-1}{q_v-1} \log q_v &= \frac{q}{q-1} \log q - \sum_{v \neq \infty} Z_v(\mathbb{1}, 1) \log q_v \\ &= \frac{q}{q-1} \log q - Z^\infty(\mathbb{1}, 0) \\ &= \frac{q}{q-1} \log q - \frac{\zeta'_A(0)}{\zeta_A(0)} \\ &= 0. \end{aligned}$$

So the value of the product $\prod_v |\int_u \omega|_v$ is 1 for the Carlitz motive.

Let us describe the structure of this article. In Section 2 we recall from [HK15] the definition of local shtukas, how to attach a local shtuka at $v \subset A$ to an A -motive \underline{M} over L with good reduction, and we discuss its cohomology realizations. In Section 3 we define the notions of complex multiplication and CM-type of a local shtuka, and in Section 4 we compute the periods and their valuations of a local shtuka with complex multiplication, and we prove Theorem 1.3. Finally in Appendix A we prove the facts used above.

2 Local Shtukas

In the rest of the article we fix a place $v \neq \infty$ of Q . We keep the notation from the introduction, except that we write $L = \kappa((\pi_L))$ for the field K_v and let $R = \kappa[[\pi_L]]$ be its valuation ring. We write $z := z_v$. Then $A_v = \mathbb{F}_v[[z]]$ and $Q_v = \mathbb{F}_v((z))$. The homomorphism $\gamma: A \rightarrow K$ extends by continuity to $\gamma: A_v \rightarrow L$ and factors through $\gamma: A_v \rightarrow R$ with $\zeta := \zeta_v = \gamma(z) \in \pi_L R \setminus \{0\}$. Let $R[[z]]$ be the power series ring in the variable z over R and $\hat{\sigma}$ the endomorphism of $R[[z]]$ with $\hat{\sigma}(z) = z$ and $\hat{\sigma}(b) = b^{q_v}$ for $b \in R$, where $q_v = \#\mathbb{F}_v$. For an $R[[z]]$ -module \hat{M} we let $\hat{\sigma}^* \hat{M} := \hat{M} \otimes_{R[[z]], \hat{\sigma}} R[[z]]$ as well as $\hat{M}[\frac{1}{z-\zeta}] := \hat{M} \otimes_{R[[z]]} R[[z]][\frac{1}{z-\zeta}]$ and $\hat{M}[\frac{1}{z}] := \hat{M} \otimes_{R[[z]]} R[[z]][\frac{1}{z}]$.

Definition 2.1. A *local $\hat{\sigma}$ -shtuka of rank r* over R is a pair $\underline{\hat{M}} = (\hat{M}, \tau_{\hat{M}})$ consisting of a free $R[[z]]$ -module \hat{M} of rank r , and an isomorphism $\tau_{\hat{M}}: \hat{\sigma}^* \hat{M}[\frac{1}{z-\zeta}] \xrightarrow{\sim} \hat{M}[\frac{1}{z-\zeta}]$. It is *effective* if $\tau_{\hat{M}}(\hat{\sigma}^* \hat{M}) \subset \hat{M}$ and *étale* if $\tau_{\hat{M}}(\hat{\sigma}^* \hat{M}) = \hat{M}$. We write $\text{rk } \underline{\hat{M}}$ for the rank of $\underline{\hat{M}}$.

A *morphism* of local shtukas $f: \underline{\hat{M}} = (\hat{M}, \tau_{\hat{M}}) \rightarrow \underline{\hat{N}} = (\hat{N}, \tau_{\hat{N}})$ over R is a morphism of the underlying modules $f: \hat{M} \rightarrow \hat{N}$ which satisfies $\tau_{\hat{N}} \circ \hat{\sigma}^* f = f \circ \tau_{\hat{M}}$. We denote the A_v -module of homomorphisms $f: \hat{M} \rightarrow \hat{N}$ by $\text{Hom}_R(\underline{\hat{M}}, \underline{\hat{N}})$ and write $\text{End}_R(\underline{\hat{M}}) = \text{Hom}_R(\underline{\hat{M}}, \underline{\hat{M}})$.

A *quasi-morphism* between local shtukas $f: (\hat{M}, \tau_{\hat{M}}) \rightarrow (\hat{N}, \tau_{\hat{N}})$ over R is a morphism of $R[[z]][\frac{1}{z}]$ -modules $f: \hat{M}[\frac{1}{z}] \xrightarrow{\sim} \hat{N}[\frac{1}{z}]$ with $\tau_{\hat{N}} \circ \hat{\sigma}^* f = f \circ \tau_{\hat{M}}$. It is called a *quasi-isogeny* if it is an isomorphism of $R[[z]][\frac{1}{z}]$ -modules. We denote the Q_v -vector space of quasi-morphisms from $\underline{\hat{M}}$ to $\underline{\hat{N}}$ by $\text{QHom}_R(\underline{\hat{M}}, \underline{\hat{N}})$ and write $\text{QEnd}_R(\underline{\hat{M}}) = \text{QHom}_R(\underline{\hat{M}}, \underline{\hat{M}})$.

Note that $\text{Hom}_R(\underline{\hat{M}}, \underline{\hat{N}})$ is a finite free A_v -module of rank at most $\text{rk } \underline{\hat{M}} \cdot \text{rk } \underline{\hat{N}}$ by [HK15, Corollary 4.5] and $\text{QHom}_R(\underline{\hat{M}}, \underline{\hat{N}}) = \text{Hom}_R(\underline{\hat{M}}, \underline{\hat{N}}) \otimes_{A_v} Q_v$. Also every quasi-isogeny $f: \underline{\hat{M}} \rightarrow \underline{\hat{N}}$ induces an isomorphism of Q_v -algebras $\text{QEnd}_R(\underline{\hat{M}}) \xrightarrow{\sim} \text{QEnd}_R(\underline{\hat{N}})$, $g \mapsto f g f^{-1}$.

Example 2.2. We assume that the A -motive $\underline{M} = (M, \tau_M)$ has *good reduction*, that is, there exist a pair $\underline{M} = (M, \tau_M)$ consisting of a locally free module M over $A_R := A \otimes_{\mathbb{F}_q} R$ of finite rank and an isomorphism $\tau_M: \sigma^* M|_{\text{Spec } A_R \setminus V(\mathcal{J})} \xrightarrow{\sim} M|_{\text{Spec } A_R \setminus V(\mathcal{J})}$ of the associated sheaves outside the vanishing locus $V(\mathcal{J}) \subset \text{Spec } A_R$ of the ideal $\mathcal{J} := (a \otimes 1 - 1 \otimes \gamma(a) : a \in A) \subset A_R$, such that $\underline{M} \otimes_R L \cong \underline{M}$. The reduction $\underline{M} \otimes_R \kappa$ is an A -motive over κ of A -characteristic $v = \ker(\gamma: A \rightarrow \kappa)$. The pair \underline{M} is called an *A-motive over R* and a *good model of \underline{M}* .

We consider the v -adic completions $A_{v,R}$ of A_R and $\underline{M} \otimes_{A_R} A_{v,R} := (M \otimes_{A_R} A_{v,R}, \tau_M \otimes \text{id})$ of \underline{M} . We let $d_v := [\mathbb{F}_v : \mathbb{F}_q]$ and discuss the two cases $d_v = 1$ and $d_v > 1$ separately. If $d_v = 1$, and hence $q_v = q$ and $\hat{\sigma} = \sigma$, we have $A_{v,R} = R[[z]]$, and $\underline{M} \otimes_{A_R} A_{v,R}$ is a local $\hat{\sigma}$ -shtuka over $\text{Spec } R$ which we denote by $\underline{\hat{M}}_v(\underline{M})$ and call the *local shtuka at v associated with \underline{M}* .

If $d_v > 1$, the situation is more complicated, because $\mathbb{F}_v \otimes_{\mathbb{F}_q} R$ and $A_{v,R}$ fail to be integral domains. Namely,

$$\mathbb{F}_v \otimes_{\mathbb{F}_q} R = \prod_{\text{Gal}(\mathbb{F}_v/\mathbb{F}_q)} \mathbb{F}_v \otimes_{\mathbb{F}_v} R = \prod_{i \in \mathbb{Z}/d_v \mathbb{Z}} \mathbb{F}_v \otimes_{\mathbb{F}_q} R / (a \otimes 1 - 1 \otimes \gamma(a)^{q^i} : a \in \mathbb{F}_v)$$

and σ transports the i -th factor to the $(i+1)$ -th factor. In particular $\hat{\sigma}$ stabilizes each factor. Denote by \mathfrak{a}_i the ideal of $A_{v,R}$ generated by $\{a \otimes 1 - 1 \otimes \gamma(a)^{q^i} : a \in \mathbb{F}_v\}$. Then

$$A_{v,R} = \prod_{\text{Gal}(\mathbb{F}_v/\mathbb{F}_q)} A_v \hat{\otimes}_{\mathbb{F}_v} R = \prod_{i \in \mathbb{Z}/d_v \mathbb{Z}} A_{v,R} / \mathfrak{a}_i.$$

Note that each factor is isomorphic to $R[[z]]$ and the ideals \mathfrak{a}_i correspond precisely to the places v_i of $C_{\mathbb{F}_v}$ lying above v . The ideal \mathcal{J} decomposes as follows $\mathcal{J} \cdot A_{v,R} / \mathfrak{a}_0 = (z - \zeta)$ and $\mathcal{J} \cdot A_{v,R} / \mathfrak{a}_i = (1)$ for $i \neq 0$. We define the *local shtuka at v associated with \underline{M}* as $\underline{\hat{M}}_v(\underline{M}) := (\hat{M}, \tau_{\hat{M}}) := (M \otimes_{A_R} A_{v,R} / \mathfrak{a}_0, (\tau_M \otimes 1)^{d_v})$,

where $\tau_{\mathcal{M}}^{d_v} := \tau_{\mathcal{M}} \circ \sigma^* \tau_{\mathcal{M}} \circ \dots \circ \sigma^{(d_v-1)*} \tau_{\mathcal{M}}$. Of course if $d_v = 1$ we get back the definition of $\hat{\underline{M}}_v(\underline{\mathcal{M}})$ given above. Also note if $\underline{\mathcal{M}}$ is effective, then $\mathcal{M}/\tau_{\mathcal{M}}(\sigma^* \mathcal{M}) = \hat{\mathcal{M}}/\tau_{\hat{\mathcal{M}}}(\hat{\sigma}^* \hat{\mathcal{M}})$.

The local shtuka $\hat{\underline{M}}_v(\underline{\mathcal{M}})$ allows to recover $\underline{\mathcal{M}} \otimes_{A_R} A_{v,R}$ via the isomorphism

$$\bigoplus_{i=0}^{d_v-1} (\tau_{\mathcal{M}} \otimes 1)^i \bmod \mathfrak{a}_i : \left(\bigoplus_{i=0}^{d_v-1} \sigma^{i*} (\mathcal{M} \otimes_{A_R} A_{v,R}/\mathfrak{a}_0), (\tau_{\mathcal{M}} \otimes 1)^{d_v} \oplus \bigoplus_{i \neq 0} \text{id} \right) \xrightarrow{\sim} \underline{\mathcal{M}} \otimes_{A_R} A_{v,R},$$

because for $i \neq 0$ the equality $\mathcal{J} \cdot A_{v,R}/\mathfrak{a}_i = (1)$ implies that $\tau_{\mathcal{M}} \otimes 1$ is an isomorphism modulo \mathfrak{a}_i ; see [BH11, Propositions 8.8 and 8.5] for more details.

Next we define the v -adic realization and the de Rham realization of a local shtuka $\hat{\underline{M}} = (\hat{M}, \tau_{\hat{M}})$ over R . Since $\tau_{\hat{M}}$ induces an isomorphism $\tau_{\hat{M}} : \hat{\sigma}^* \hat{M} \otimes_{R[[z]]} L[[z]] \xrightarrow{\sim} \hat{M} \otimes_{R[[z]]} L[[z]]$, we can think of $\hat{\underline{M}} \otimes_{R[[z]]} L[[z]]$ as an étale local shtuka over L .

Definition 2.3. The v -adic realization $H_v^1(\hat{\underline{M}}, A_v)$ of a local $\hat{\sigma}$ -shtuka $\hat{\underline{M}} = (\hat{M}, \tau_{\hat{M}})$ is the $\mathcal{G}_L := \text{Gal}(L^{\text{sep}}/L)$ -module of τ -invariants

$$H_v^1(\hat{\underline{M}}, A_v) := (\hat{M} \otimes_{R[[z]]} L^{\text{sep}}[[z]])^{\tau} := \{m \in \hat{M} \otimes_{R[[z]]} L^{\text{sep}}[[z]] : \tau_{\hat{M}}(\hat{\sigma}_M^* m) = m\},$$

where we set $\hat{\sigma}_M^* m := m \otimes 1 \in \hat{M} \otimes_{R[[z]], \hat{\sigma}} R[[z]] =: \sigma^* M$ for $m \in M$. One also writes sometimes $\check{T}_v \hat{\underline{M}} = H_v^1(\hat{\underline{M}}, A_v)$ and calls this the *dual Tate module of $\hat{\underline{M}}$* . By [HK15, Proposition 4.2] it is a free A_v -module of the same rank as \hat{M} . We also write $H_v^1(\hat{\underline{M}}, B) := H_v^1(\hat{\underline{M}}, A_v) \otimes_{A_v} B$ for an A_v -algebra B .

If $\underline{M} = (M, \tau_M)$ is an A -motive over L with good model $\underline{\mathcal{M}}$ and $\hat{\underline{M}} = \hat{\underline{M}}_v(\underline{\mathcal{M}})$ is the local shtuka at v associated with $\underline{\mathcal{M}}$, then $H_v^1(\hat{\underline{M}}, A_v)$ is by [HK15, Proposition 4.6] canonically isomorphic as a representation of \mathcal{G}_L to the v -adic realization of $\underline{\mathcal{M}}$, which is defined as

$$H_v^1(\underline{\mathcal{M}}, A_v) := \{m \in M \otimes_{A_L} A_{v,L^{\text{sep}}} : \tau_M(\sigma_M^* m) = m\},$$

where we set $\sigma_M^* m := m \otimes 1 \in M \otimes_{A_R, \sigma} A_R =: \sigma^* M$ for $m \in M$ and where $A_{v,L^{\text{sep}}}$ is the v -adic completion of $A_{L^{\text{sep}}}$.

Definition 2.4. Let $\hat{\underline{M}} = (\hat{M}, \tau_{\hat{M}})$ be a local $\hat{\sigma}$ -shtuka over R . We define the *de Rham realizations* of $\hat{\underline{M}}$ as

$$\begin{aligned} H_{\text{dR}}^1(\hat{\underline{M}}, R) &:= \hat{\sigma}^* \hat{M} / (z - \zeta) \hat{M} = \hat{\sigma}^* \hat{M} \otimes_{R[[z]], z \mapsto \zeta} R, \quad \text{as well as} \\ H_{\text{dR}}^1(\hat{\underline{M}}, L[[z - \zeta]]) &:= \hat{\sigma}^* \hat{M} \otimes_{L[[z]]} L[[z - \zeta]] \quad \text{and} \\ H_{\text{dR}}^1(\hat{\underline{M}}, L) &:= \hat{\sigma}^* \hat{M} \otimes_{R[[z]], z \mapsto \zeta} L = H_{\text{dR}}^1(\hat{\underline{M}}, L[[z - \zeta]]) \otimes_{L[[z - \zeta]]} L[[z - \zeta]] / (z - \zeta) \\ &= H_{\text{dR}}^1(\hat{\underline{M}}, R) \otimes_R L. \end{aligned}$$

It carries the *Hodge-Pink lattice* $\mathfrak{q}^{\hat{\underline{M}}} := \tau_{\hat{M}}^{-1}(\hat{M} \otimes_{R[[z]]} L[[z - \zeta]]) \subset H_{\text{dR}}^1(\hat{\underline{M}}, L[[z - \zeta]])[\frac{1}{z - \zeta}]$. We also write $H_{\text{dR}}^1(\hat{\underline{M}}, B) := H_{\text{dR}}^1(\hat{\underline{M}}, L[[z - \zeta]]) \otimes_{L[[z - \zeta]]} B$ for an $L[[z - \zeta]]$ -algebra B .

If $\underline{M} = (M, \tau_M)$ is an A -motive over L with good model $\underline{\mathcal{M}}$ and $\hat{\underline{M}} = \hat{\underline{M}}_v(\underline{\mathcal{M}})$ is the local shtuka at v associated with $\underline{\mathcal{M}}$ and $d_v = [\mathbb{F}_v : \mathbb{F}_q]$ is as in Example 2.2, the map

$$\sigma^* \tau_M^{d_v-1} = \sigma^* \tau_M \circ \sigma^{2*} \tau_M \circ \dots \circ \sigma^{(d_v-1)*} \tau_M : \sigma^{d_v*} M \otimes_{A_R} A_{v,R}/\mathfrak{a}_0 \xrightarrow{\sim} \sigma^* M \otimes_{A_R} A_{v,R}/\mathfrak{a}_0$$

is an isomorphism, because τ_M is an isomorphism over $A_{v,R}/\mathfrak{a}_i$ for all $i \neq 0$. Therefore it defines canonical isomorphisms of the de Rham realizations

$$\begin{aligned} \sigma^* \tau_M^{d_v-1} : H_{\text{dR}}^1(\hat{\underline{M}}, L[[z - \zeta]]) &\xrightarrow{\sim} H_{\text{dR}}^1(\underline{\mathcal{M}}, L[[z - \zeta]]) \quad \text{and} \\ \sigma^* \tau_M^{d_v-1} : H_{\text{dR}}^1(\hat{\underline{M}}, L) &\xrightarrow{\sim} H_{\text{dR}}^1(\underline{\mathcal{M}}, L), \end{aligned}$$

which are compatible with the Hodge-Pink lattices.

Remark 2.5. By [HK15, Theorem 4.14] there is a canonical comparison isomorphism

$$h_{v,\text{dR}}: H_v^1(\hat{\underline{M}}, \mathbb{C}_v((z - \zeta))) \xrightarrow{\sim} H_{\text{dR}}^1(\hat{\underline{M}}, \mathbb{C}_v((z - \zeta))) \quad (2.1)$$

which is equivariant for the action of \mathcal{G}_L . For our computations below we need an explicit description of $h_{v,\text{dR}}$. It is constructed as follows. The natural inclusion $H_v^1(\hat{\underline{M}}, A_v) \subset \hat{M} \otimes_{R[[z]]} L^{\text{sep}}[[z]]$ defines a canonical isomorphism of $L^{\text{sep}}[[z]]$ -modules

$$H_v^1(\hat{\underline{M}}, A_v) \otimes_{A_v} L^{\text{sep}}[[z]] \xrightarrow{\sim} \hat{M} \otimes_{R[[z]]} L^{\text{sep}}[[z]], \quad (2.2)$$

which is \mathcal{G}_L and τ -equivariant, where on the left module \mathcal{G}_L acts on both factors and τ is $\text{id} \otimes \hat{\sigma}$ and on the right module \mathcal{G}_L acts only on $L^{\text{sep}}[[z]]$ and τ is $(\tau_{\hat{M}} \circ \hat{\sigma}_{\hat{M}}^*) \otimes \hat{\sigma}$. Since $(L^{\text{sep}})^{\mathcal{G}_L} = L$ we obtain

$$\hat{M} \otimes_{R[[z]]} L[[z]] = (H_v^1(\hat{\underline{M}}, A_v) \otimes_{A_v} L^{\text{sep}}[[z]])^{\mathcal{G}_L}.$$

It turns out, see [HK15, Remark 4.3], that the isomorphism (2.2) extends to an equivariant isomorphism

$$h: H_v^1(\hat{\underline{M}}, A_v) \otimes_{A_v} L^{\text{sep}}\langle \frac{z}{\zeta} \rangle \xrightarrow{\sim} \hat{M} \otimes_{R[[z]]} L^{\text{sep}}\langle \frac{z}{\zeta} \rangle, \quad (2.3)$$

where for an $r \in \mathbb{R}_{>0}$ we use the notation

$$L^{\text{sep}}\langle \frac{z}{\zeta^r} \rangle := \left\{ \sum_{i=0}^{\infty} b_i z^i : b_i \in L^{\text{sep}}, |b_i| |\zeta|^{ri} \rightarrow 0 \ (i \rightarrow +\infty) \right\}.$$

These are subrings of $L^{\text{sep}}[[z]]$ and the endomorphism $\hat{\sigma}: \sum_i b_i z^i \mapsto \sum_i b_i^{q_v} z^i$ of $L^{\text{sep}}[[z]]$ restricts to a homomorphism $\hat{\sigma}: L^{\text{sep}}\langle \frac{z}{\zeta^r} \rangle \rightarrow L^{\text{sep}}\langle \frac{z}{\zeta^{rq_v}} \rangle$. Note that the τ -equivariance of h means $h \otimes \text{id}_{L^{\text{sep}}\langle \frac{z}{\zeta^{q_v}} \rangle} = \tau_{\hat{M}} \circ \hat{\sigma}^* h$. Now the period isomorphism is defined as

$$\begin{aligned} h_{v,\text{dR}} &:= (\tau_{\hat{M}}^{-1} \circ h) \otimes \text{id}_{\mathbb{C}_v((z - \zeta))}: H_v^1(\hat{\underline{M}}, \mathbb{C}_v((z - \zeta))) \xrightarrow{\sim} \hat{\sigma}^* \hat{M} \otimes_{R[[z]]} \mathbb{C}_v((z - \zeta)) \\ &= H_{\text{dR}}^1(\hat{\underline{M}}, \mathbb{C}_v((z - \zeta))). \end{aligned} \quad (2.4)$$

3 Local Shtukas with Complex Multiplication

Definition 3.1. Let $\hat{\underline{M}}$ be a local $\hat{\sigma}$ -shtuka over R and assume that there is a commutative, semi-simple Q_v -algebra $E_v \subset \text{QEnd}_R(\hat{\underline{M}}) := \text{End}_R(\hat{\underline{M}}) \otimes_{A_v} Q_v$ with $\dim_{Q_v} E_v = \text{rk } \hat{\underline{M}}$. Then we say that $\hat{\underline{M}}$ has *complex multiplication* (by E_v).

Here again semi-simple means that E_v is a direct product $E_v = E_{v,1} \times \cdots \times E_{v,s}$ of finite field extensions of Q_v . We do *not* assume that E_v is itself a field and in Section 3 we do *not* assume that the $E_{v,i}$ are separable over Q_v . We let \mathcal{O}_{E_v} be the integral closure of A_v in E_v . It is a product $\mathcal{O}_{E_v} = \mathcal{O}_{E_{v,1}} \times \cdots \times \mathcal{O}_{E_{v,s}}$ of complete discrete valuation rings where $\mathcal{O}_{E_{v,i}}$ is the integral closure of A_v in the field $E_{v,i}$. For every i we write $\mathcal{O}_{E_{v,i}} = \mathbb{F}_{\tilde{v}_i}[[y_i]]$ and set $f_i := [\mathbb{F}_{\tilde{v}_i} : \mathbb{F}_v]$ and $e_i := \text{ord}_{y_i}(z)$. Then $f_i e_i = [E_{v,i} : Q_v]$ and e_i is divisible by the inseparability degree of $E_{v,i}$ over Q_v . Also we write $\tilde{q}_i := \#\mathbb{F}_{\tilde{v}_i} = q_v^{f_i}$.

Proposition 3.2. *There is a local shtuka $\hat{\underline{M}}'$ over R quasi-isogenous to $\hat{\underline{M}}$ with $\mathcal{O}_{E_v} \subset \text{End}_R(\hat{\underline{M}}')$.*

Proof. The A_v -submodule $T' := \mathcal{O}_{E_v} \cdot H_v^1(\hat{\underline{M}}, A_v) \subset H_v^1(\hat{\underline{M}}, Q_v)$ is \mathcal{G}_L -invariant and contains $H_v^1(\hat{\underline{M}}, A_v)$. Since $\mathcal{O}_{E_v} \subset \text{QEnd}_R(\hat{\underline{M}}) = \text{End}_R(\hat{\underline{M}}) \otimes_{A_v} Q_v$ there is an element $a \in A_v$ with $a \cdot \mathcal{O}_{E_v} \subset \text{End}_R(\hat{\underline{M}})$, and therefore $a \cdot T' \subset H_v^1(\hat{\underline{M}}, A_v)$ is a finitely generated A_v -module, that is an A_v -lattice. By [HK15, Proposition 4.19] there is a local shtuka $\hat{\underline{M}}'$ and a quasi-isogeny $f: \hat{\underline{M}} \rightarrow \hat{\underline{M}}'$ which maps T' isomorphically onto $H_v^1(\hat{\underline{M}}', A_v)$. In particular \mathcal{O}_{E_v} acts as \mathcal{G}_L -equivariant endomorphisms of $H_v^1(\hat{\underline{M}}', A_v)$. Since the functor $\hat{\underline{M}}' \mapsto H_v^1(\hat{\underline{M}}', A_v)$ from local shtukas to $A_v[\mathcal{G}_L]$ -modules is fully faithful by [HK15, Theorem 4.17], we see that $\mathcal{O}_{E_v} \subset \text{End}_R(\hat{\underline{M}}')$. \square

Definition 3.3. If $\mathcal{O}_{E_v} \subset \text{End}_R(\hat{M})$ we say that \hat{M} has *complex multiplication by \mathcal{O}_{E_v}* . By the proposition we may from now on assume that this is the case. It makes the underlying module \hat{M} into a module over the ring $\mathcal{O}_{E_v, R} := \mathcal{O}_{E_v} \otimes_{A_v} R[[z]] = \mathcal{O}_{E_v} \hat{\otimes}_{\mathbb{F}_v} R$. For $a \in \mathcal{O}_{E_v}$ note that $a \otimes 1 \in \mathcal{O}_{E_v, R}$ acts on \hat{M} as the endomorphism a and on $\hat{\sigma}^* \hat{M}$ as the endomorphism $\hat{\sigma}^* a$ and $\tau_{\hat{M}}$ is \mathcal{O}_{E_v} -linear because $a \circ \tau_{\hat{M}} = \tau_{\hat{M}} \circ \hat{\sigma}^* a$.

3.4. Let us assume that L contains $\psi(E_v)$ for every $\psi \in H_{E_v}$. This implies $\psi(\mathcal{O}_{E_v}) \subset R$ for every $\psi \in H_{E_v}$. Under this assumption let us describe the ring $\mathcal{O}_{E_v, R} = \prod_{i=1}^s \mathcal{O}_{E_v, i, R}$. Fix an i and choose and fix an \mathbb{F}_v -homomorphism $\mathbb{F}_{\tilde{v}_i} \hookrightarrow \kappa$. Then $H_{\tilde{v}_i} := \text{Hom}_{\mathbb{F}_v}(\mathbb{F}_{\tilde{v}_i}, \kappa) \cong \mathbb{Z}/f_i \mathbb{Z}$ under the map that sends $j \in \mathbb{Z}/f_i \mathbb{Z}$ to the homomorphism $(\lambda \mapsto \lambda^{q_v^j}) \in H_{\tilde{v}_i}$. Also

$$\mathbb{F}_{\tilde{v}_i} \otimes_{\mathbb{F}_v} R = \prod_{H_{\tilde{v}_i}} R = \prod_{j \in \mathbb{Z}/f_i \mathbb{Z}} \mathbb{F}_{\tilde{v}_i} \otimes_{\mathbb{F}_v} R / (\lambda \otimes 1 - 1 \otimes \lambda^{q_v^j} : \lambda \in \mathbb{F}_{\tilde{v}_i})$$

and $\hat{\sigma}^*$ transports the j -th factor to the $(j+1)$ -th factor. Denote by $\mathfrak{b}_{i,j} \subset \mathcal{O}_{E_v, i}$ the ideal generated by $(\lambda \otimes 1 - 1 \otimes \lambda^{q_v^j} : \lambda \in \mathbb{F}_{\tilde{v}_i})$. Then

$$\mathcal{O}_{E_v, i, R} := \mathbb{F}_{\tilde{v}_i}[[y_i]] \otimes_{A_v} R[[z]] = \prod_{j \in \mathbb{Z}/f_i \mathbb{Z}} \mathbb{F}_{\tilde{v}_i}[[y_i]] \otimes_{\mathbb{F}_v[[z]]} R[[z]] / \mathfrak{b}_{i,j} = \prod_{H_{\tilde{v}_i}} R[[y_i]]. \quad (3.1)$$

For every $\psi \in H_{E_v}$ we let $i(\psi)$ be such that ψ factors through the quotient $E_v \twoheadrightarrow E_{v, i(\psi)}$ and we let $j(\psi) \in \mathbb{Z}/f_{i(\psi)} \mathbb{Z}$ be the element such that $\psi(\lambda) = \lambda^{q_v^{j(\psi)}}$ for all $\lambda \in \mathbb{F}_{\tilde{v}_{i(\psi)}}$. Then the morphism $\psi: \mathcal{O}_{E_v} \rightarrow R$ equals the composition $\mathcal{O}_{E_v} \hookrightarrow \mathcal{O}_{E_v, R} \twoheadrightarrow \mathcal{O}_{E_v, i(\psi), R} / (\mathfrak{b}_{i(\psi), j(\psi)}, y_{i(\psi)})$ and $H_{E_v, i} = \{\psi \in H_{E_v} : i(\psi) = i\}$.

Lemma 3.5. *Let p^m be the inseparability degree of $E_{v, i}$ over Q_v . Then in the j -th component $R[[y_i]]$ of (3.1) we have*

$$z - \zeta = \varepsilon \cdot \prod_{\psi \in H_{E_v} : (i, j)(\psi) = (i, j)} (y_i - \psi(y_i))^{p^m} \quad (3.2)$$

for a unit $\varepsilon \in R[[y_i]]$.

Proof. Set $y'_i := y_i^{p^m}$ and let $P = P(z, Y) = \sum_{\mu, \nu} b_{\mu\nu} z^\mu Y^\nu \in \mathbb{F}_{\tilde{v}_i}[[z]][Y]$ with $b_{\mu\nu} \in \mathbb{F}_{\tilde{v}_i}$ be the minimal polynomial of y'_i over $\mathbb{F}_{\tilde{v}_i}((z))$. It is an Eisenstein polynomial of degree e_i/p^m , because $\mathbb{F}_{\tilde{v}_i}((y'_i))$ is purely ramified and separable over $\mathbb{F}_{\tilde{v}_i}((z))$ by Lemma A.2 in the appendix. In particular $b_{0, \nu} = 0$ for $0 \leq \nu < e_i/p^m$, and $b_{1,0} \neq 0$. Consider the polynomials $P^{(j)}(z, Y) := \sum_{\mu, \nu} b_{\mu\nu}^{q_v^j} z^\mu Y^\nu \in \mathbb{F}_{\tilde{v}_i}[[z]][Y] \subset R[[z]][Y]$ and $P^{(j)}(\zeta, Y) \in R[Y]$. If $\psi \in H_{E_v}$ satisfies $(i, j)(\psi) = (i, j)$ then $P^{(j)}(\zeta, \psi(y'_i)) = \psi(P(z, y'_i)) = \psi(0) = 0$. These zeroes $\psi(y'_i)$ of $P^{(j)}(\zeta, Y)$ in L are pairwise different, because if $\psi(y'_i) = \tilde{\psi}(y'_i)$ then $(i, j)(\psi) = (i, j)(\tilde{\psi})$ implies that ψ and $\tilde{\psi}$ coincide on $E_{v, i}$ and hence on E_v . It follows that $P^{(j)}(\zeta, Y) = \prod_{\psi: (i, j)(\psi) = (i, j)} (Y - \psi(y'_i))$ in $L[Y]$, whence already in $R[Y]$. In the j -th component $R[[y_i]]$ of (3.1) we have $0 = \sum_{\mu, \nu} b_{\mu\nu} z^\mu (y'_i)^\nu \otimes 1 = \sum_{\mu, \nu} (y'_i)^\nu \otimes b_{\mu\nu}^{q_v^j} z^\mu = P^{(j)}(z, y'_i)$, and we compute

$$\begin{aligned} \prod_{\psi: (i, j)(\psi) = (i, j)} (y_i - \psi(y_i))^{p^m} &= P^{(j)}(\zeta, y'_i) \\ &= P^{(j)}(\zeta, y'_i) - P^{(j)}(z, y'_i) \\ &= \sum_{\mu, \nu} b_{\mu\nu}^{q_v^j} (\zeta^\mu - z^\mu) (y'_i)^\nu \\ &= (\zeta - z) \cdot \sum_{\mu, \nu} b_{\mu\nu}^{q_v^j} (\zeta^{\mu-1} + \zeta^{\mu-2} z + \dots + z^{\mu-1}) (y'_i)^\nu. \end{aligned}$$

The factor $\sum_{\mu,\nu} b_{\mu\nu}^{q_v^j} (\zeta^{\mu-1} + \zeta^{\mu-2}z + \dots + z^{\mu-1})(y'_i)^\nu$ is congruent to $b_{1,0}^{q_v^j} \neq 0$ modulo the maximal ideal $(\pi_L, y_i) \subset R[[y_i]]$ and therefore a unit in $R[[y_i]]$. This finishes the proof. Note that since

$$P^{(j)}(\zeta, y'_i) - P^{(j)}(z, y'_i) = \sum_{n \geq 1} \frac{1}{n!} \frac{\partial^n P^{(j)}}{\partial z^n}(z, y'_i) \cdot (\zeta - z)^n,$$

the proof could also be phrased by saying that $\frac{\partial P^{(j)}}{\partial z}(z, y'_i) = \sum_{\mu,\nu} \mu b_{\mu\nu}^{q_v^j} z^{\mu-1} (y'_i)^\nu$ lies in $\mathcal{O}_{E'_{v,i}}^\times$. In fact this partial derivative is congruent to $b_{1,0}^{q_v^j} \neq 0$ modulo $y'_i \cdot \mathcal{O}_{E'_{v,i}}$. \square

Let us draw a direct corollary from the proof of this lemma. To formulate it, recall that if $E_{v,i}$ is separable over Q_v , the different $\mathfrak{D}_{E_{v,i}/Q_v}$ of $E_{v,i}$ over Q_v is defined as the ideal in $\mathcal{O}_{E_{v,i}}$ which annihilates the module $\Omega_{\mathcal{O}_{E_{v,i}}/A_v}^1$ of relative differentials.

Corollary 3.6. *If $E_{v,i}$ is separable over Q_v then $\mathfrak{D}_{\varphi(E_{v,i})/Q_v} = \left(\frac{z-\zeta}{y_i-\varphi(y_i)} \Big|_{y_i=\varphi(y_i)} \right)$ in $\mathcal{O}_{\varphi(E_{v,i})}$.*

Proof. By [Ser79, Chapter III, §4, Proposition 8] the different is multiplicative, that is $\mathfrak{D}_{E_{v,i}/Q_v} = \mathfrak{D}_{E_{v,i}/\mathbb{F}_{\tilde{v}_i}((z))} \cdot \mathfrak{D}_{\mathbb{F}_{\tilde{v}_i}((z))/Q_v}$. Moreover, $\mathfrak{D}_{\mathbb{F}_{\tilde{v}_i}((z))/Q_v} = 1$ because $\mathbb{F}_{\tilde{v}_i}[[z]]$ is unramified over A_v . As in the proof of the preceding lemma let $P(z, Y)$ be the minimal polynomial of y'_i over $\mathbb{F}_{\tilde{v}_i}((z))$ and note that $y'_i = y_i$ under our separability assumption. Then $\frac{\partial P}{\partial z}(z, y_i) \in \mathcal{O}_{E_{v,i}}^\times$ and

$$\Omega_{\mathcal{O}_{E_{v,i}}/\mathbb{F}_{\tilde{v}_i}[[z]]}^1 = \mathcal{O}_{E_{v,i}} \langle dz, dy_i \rangle / (dz, \frac{\partial P}{\partial z}(z, y_i) dz + \frac{\partial P}{\partial Y}(z, y_i) dy_i) = \mathcal{O}_{E_{v,i}} \cdot dy_i / (\frac{\partial P}{\partial Y}(z, y_i) dy_i).$$

We write $z = f(y_i) \in \mathbb{F}_{\tilde{v}_i}[[y_i]]$. Then $0 = \frac{d}{dy_i} P(f(y_i), y_i) = \frac{\partial P}{\partial z}(f(y_i), y_i) \frac{df(y_i)}{dy_i} + \frac{\partial P}{\partial Y}(f(y_i), y_i)$ and $\mathfrak{D}_{E_{v,i}/Q_v} = \left(\frac{\partial P}{\partial Y}(z, y_i) \right) = \left(\frac{df(y_i)}{dy_i} \right)$. Now Lemma A.1 in the appendix implies that $\mathfrak{D}_{\varphi(E_{v,i})/Q_v} = \varphi\left(\frac{df(y_i)}{dy_i}\right) = \left(\frac{z-\zeta}{y_i-\varphi(y_i)} \right) \Big|_{y_i=\varphi(y_i)}$. \square

Now we explore the consequences of these decompositions for local shtukas with complex multiplication.

Proposition 3.7. *Let $\hat{M} = (\hat{M}, \tau_{\hat{M}})$ have complex multiplication by \mathcal{O}_{E_v} . Then the $\mathcal{O}_{E_v, R}$ -module \hat{M} is free of rank 1. In particular, $\hat{M}_i := \hat{M} \otimes_{\mathcal{O}_{E_v}} \mathcal{O}_{E_{v,i}}$ is a local $\hat{\sigma}$ -shtuka over R with $\text{rk } \hat{M}_i = [E_{v,i} : Q_v]$ and $\hat{M} = \bigoplus_{i=1}^s \hat{M}_i$.*

Proof. By faithfully flat descent [EGA, IV₂, Proposition 2.5.2], we may replace R by a finite extension of discrete valuation rings. Therefore it suffices to prove the proposition in the case where R contains $\psi(\mathcal{O}_{E_v})$ for all $\psi \in H_{E_v}$. In this case $\mathcal{O}_{E_v, R}$ is a product of two dimensional regular local rings $R[[y_i]]$ by (3.1). By [Ser58, §6, Lemme 6] a finitely generated module M over such a ring is free if and only if it is reflexive, that is M is isomorphic to its bidual $M^{\vee\vee}$, where $M^\vee = \text{Hom}_{R[[y_i]]}(M, R[[y_i]])$. In particular $M^{\vee\vee}$, which is isomorphic to $(M^{\vee\vee})^{\vee\vee}$ is free. We apply this to $M := \hat{M} \otimes_{\mathcal{O}_{E_v, R}} R[[y_i]]$ and consider the base changes $M \otimes_{R[[y_i]]} L[[y_i]] = M \otimes_{R[[z]]} L[[z]]$ and $M \otimes_{R[[y_i]]} R[[y_i]][\frac{1}{y_i}] = M \otimes_{R[[z]]} R[[z]][\frac{1}{z}]$. Like $L[[y_i]]$ also $R[[y_i]][\frac{1}{y_i}]$ is a principal ideal domain, because it is a factorial ring of dimension 1. Using [Eis95, Proposition 2.10] and that both base changes of M are torsion free, whence free, we see that the canonical morphism $M \rightarrow M^{\vee\vee}$ is an isomorphism after both base changes. Since $R[[z]] = L[[z]] \cap R[[z]][\frac{1}{z}] \subset L((z))$ and M and $M^{\vee\vee}$ are free $R[[z]]$ -modules, M equals the intersection $(M \otimes_{R[[z]]} L[[z]]) \cap (M \otimes_{R[[z]]} R[[z]][\frac{1}{z}])$ inside $M \otimes_{R[[z]]} L((z))$, and likewise for $M^{\vee\vee}$. This shows that $M \rightarrow M^{\vee\vee}$ is an isomorphism and M is free over $R[[y_i]]$.

It remains to compute the rank. Let $r_{i,j} := \text{rk}_{R[[y_i]]}(\hat{M} \otimes_{\mathcal{O}_{E_v, R}} \mathcal{O}_{E_{v,i}, R}/\mathfrak{b}_{i,j})$ for all $i = 1, \dots, s$ and all $j \in \mathbb{Z}/f_i\mathbb{Z}$. Then $\sum_{i,j} r_{i,j} \cdot e_i = \text{rk } \hat{M}$. We first prove that for a fixed i all $r_{i,j}$ are equal. Since

$(\hat{\sigma}^* \hat{M}) \otimes_{\mathcal{O}_{E_v, R}} \mathcal{O}_{E_v, i, R} / \mathfrak{b}_{i, j} = \hat{\sigma}^*(\hat{M} \otimes_{\mathcal{O}_{E_v, R}} \mathcal{O}_{E_v, i, R} / \mathfrak{b}_{i, j-1}) \cong R[[y_i]]^{r_{i, j-1}}$, we can write the isomorphism $\tau_{\hat{M}} : \hat{\sigma}^* \hat{M}[\frac{1}{z-\zeta}] \xrightarrow{\sim} \hat{M}[\frac{1}{z-\zeta}]$ in the form

$$\prod_j R[[y_i]][\frac{1}{z-\zeta}]^{r_{i, j-1}} \xrightarrow{\sim} \prod_j R[[y_i]][\frac{1}{z-\zeta}]^{r_{i, j}},$$

which gives us $r_{i, j-1} = r_{i, j} =: r_i$ for all j , and hence $\sum_i r_i f_i e_i = \text{rk } \hat{M} = \dim_{Q_v} E_v = \sum_i \dim_{Q_v} E_{v, i} = \sum_i f_i e_i$. Thus if we prove that $r_i \neq 0$ then all r_i must be 1 and so \hat{M} is a free $\mathcal{O}_{E_v, R}$ -module of rank 1 and $\text{rk } \hat{M}_i = f_i e_i = [E_{v, i} : Q_v]$. Now $r_i = 0$ means that $\hat{M} \otimes_{\mathcal{O}_{E_v}} \mathcal{O}_{E_v, i} = (0)$, and hence $E_{v, i}$ acts as zero on \hat{M} in contradiction to $E_v \subset \text{QEnd}_R(\hat{M})$. This finishes the proof. \square

Proposition 3.8. *If \hat{M} has complex multiplication by a commutative semi-simple Q_v -algebra E_v then $H_v^1(\hat{M}, Q_v)$ is a free E_v -module of rank 1 and $H_{\text{dR}}^1(\hat{M}, L[[z - \zeta]])$ is a free $E_v \otimes_{Q_v} L[[z - \zeta]]$ -module of rank one, where the homomorphism $Q_v = \mathbb{F}_v((z)) \rightarrow L[[z - \zeta]]$ is given by $z \mapsto z = \zeta + (z - \zeta)$. If we assume that $L \supset \psi(E_v)$ for all $\psi \in H_{E_v}$ then the decomposition (A.1) induces a decomposition*

$$H_{\text{dR}}^1(\hat{M}, L[[z - \zeta]]) = \bigoplus_{\psi \in H_{E_v}} H^\psi(\hat{M}, L[[y_{i(\psi)} - \psi(y_{i(\psi)})]]), \quad (3.3)$$

where $H^\psi(\hat{M}, L[[y_{i(\psi)} - \psi(y_{i(\psi)})]])$ is free of rank 1 over $L[[y_{i(\psi)} - \psi(y_{i(\psi)})]]$. In particular,

$$H_{\text{dR}}^1(\hat{M}, L) = \bigoplus_{\psi \in H_{E_v}} H^\psi(\hat{M}, R) \otimes_R L \quad (3.4)$$

is the decomposition into generalized eigenspaces of the E_v -action. Here

$$H^\psi(\hat{M}, R) := \{ \omega \in H_{\text{dR}}^1(\hat{M}, R) : ([a]^* - \psi(a))^n \cdot \omega = 0 \ \forall n \gg 0 \text{ and } a \in E_v \cap \text{End}_R(\hat{M}) \}$$

is a free R -module of rank equal to the inseparability degree of $E_{v, i}$ over Q_v .

Proof. By the faithfulness of the functor $\hat{M} \rightarrow H_v^1(\hat{M}, Q_v)$ we have $E_v \subset \text{End}_{Q_v} H_v^1(\hat{M}, Q_v)$. So the first statement follows from [BH09, Lemma 7.2].

Since $H_{\text{dR}}^1(\hat{M}, L[[z - \zeta]])$ is an isogeny invariant, we may by Proposition 3.2 assume that $\mathcal{O}_{E_v} \subset \text{End}_R(\hat{M})$ and then \hat{M} is free of rank 1 over $\mathcal{O}_{E_v, R}$ by Proposition 3.7. It follows that $H_{\text{dR}}^1(\hat{M}, L[[z - \zeta]]) := \hat{\sigma}^* \hat{M} \otimes_{R[[z]]} L[[z - \zeta]] \cong E_v \otimes_{Q_v} L[[z - \zeta]]$. Now we use Lemma A.3. In particular, (3.4) and the statement about $H^\psi(\hat{M}, R)$ follow from (A.2) and the equation $H_{\text{dR}}^1(\hat{M}, L) = H_{\text{dR}}^1(\hat{M}, R) \otimes_R L$. \square

The proposition allows us to make two definitions.

Definition 3.9. Let \hat{M} have complex multiplication by \mathcal{O}_{E_v} and assume that $L \supset \psi(E_v)$ for all $\psi \in H_{E_v}$. Fix a $\psi \in H_{E_v}$ and let $i := i(\psi)$. Let $\omega_\psi^\circ \in H^\psi(\hat{M}, L[[y_i - \psi(y_i)]])$ be an $L[[y_i - \psi(y_i)]]$ -generator whose reduction $\omega_\psi^\circ \bmod (y_i - \psi(y_i)) \in H^\psi(\hat{M}, L) / (y_i - \psi(y_i)) H^\psi(\hat{M}, L)$ is a generator of the free R -module of rank one $H^\psi(\hat{M}, R) / (y_i - \psi(y_i)) H^\psi(\hat{M}, R)$. Such a ω_ψ° is uniquely determined up to multiplication by an element of $R^\times + (y_i - \psi(y_i)) L[[y_i - \psi(y_i)]]$. Note that if $E_{v, i}$ is separable over Q_v then $y_i - \psi(y_i)$ acts trivially on $H^\psi(\hat{M}, L)$ and $H^\psi(\hat{M}, R)$ is a free R -module of rank 1. Also $L[[y_i - \varphi(y_i)]] = L[[z - \zeta]]$.

If $\omega_\psi \in H^\psi(\hat{M}, L[[y_i - \psi(y_i)]])$ is any generator, there is an element $x \in L[[y_i - \psi(y_i)]]^\times$ with $\omega_\psi = x \omega_\psi^\circ$. We define the *valuation* of ω_ψ as $v(\omega_\psi) := v(x \bmod y_i - \psi(y_i))$. It only depends on the image of ω_ψ in $H_{\text{dR}}^1(\hat{M}, L)$ and is also independent of the choice of ω_ψ° .

Note that if $\underline{M} = (M, \tau_M)$ is an A -motive over L with good model \underline{M} over R , and $\hat{M} = \hat{M}_v(\underline{M})$ is the local shtuka at v associated with \underline{M} as in Example 2.2, then for an $L[[y_i - \psi(y_i)]]$ -generator $\omega_\psi \in H^\psi(\underline{M}, L[[y_i - \psi(y_i)]]) = H^\psi(\hat{M}, L[[y_i - \psi(y_i)]])$ the present definition of $v(\omega_\psi)$ coincides with the definition of $v(\omega_\psi)$ from (1.13).

Definition 3.10. A *local CM-type* at v is a pair (E_v, Φ) with E_v a semisimple commutative Q_v -algebra and $\Phi = (d_\psi)_{\psi \in H_{E_v}}$ a tuple of integers $d_\psi \in \mathbb{Z}$.

If \hat{M} is a local shtuka with complex multiplication by a commutative semi-simple Q_v -algebra E_v and if $L \supset \psi(E_v)$ for all $\psi \in H_{E_v}$ then the Hodge-Pink lattice $\mathfrak{q}^{\hat{M}} = \tau_M^{-1}(\hat{M} \otimes_{R[[z]]} L[[z - \zeta]])$ of \hat{M} satisfies $\mathfrak{q}^{\hat{M}} = \prod_{\psi \in H_{E_v}} (y_{i(\psi)} - \psi(y_{i(\psi)}))^{-d_\psi} H^\psi(\hat{M}, L[[y_{i(\psi)} - \psi(y_{i(\psi)})]])$ for integers d_ψ under the decomposition (3.3). We call $\Phi = (d_\psi)_{\psi \in H_{E_v}}$ the *local CM-type* of \hat{M} .

Note that if $\underline{M} = (M, \tau_M)$ is an A -motive over L with good model \underline{M} over R , and $\hat{M} = \hat{M}_v(\underline{M})$ is the local shtuka at v associated with \underline{M} as in Example 2.2, and $E_v := E \otimes_Q Q_v$, then we see from the isomorphism $H^\psi(\underline{M}, L[[y_{i(\psi)} - \psi(y_{i(\psi)})]]) = H^\psi(\hat{M}, L[[y_{i(\psi)} - \psi(y_{i(\psi)})]])$ that the local CM-type of \hat{M} is equal to the CM-type of \underline{M} under the identification $H_E \xrightarrow{\sim} H_{E_v}$, which extends $\psi: E \rightarrow Q_v^{\text{alg}} \subset Q_v^{\text{alg}}$ to the completion $\psi: E_v \rightarrow Q_v^{\text{alg}}$.

4 Periods of Local Shtukas with Complex Multiplication

4.1. In this section we let \hat{M} be a local $\hat{\sigma}$ -shtuka over R with complex multiplication by \mathcal{O}_{E_v} where E_v is a commutative semi-simple Q_v -algebra as in the preceding section. From Theorem 4.13 on we assume that the factors $E_{v,i}$ of E_v are *separable* field extensions of Q_v . Throughout we assume that $L \supset \psi(E_v)$ for all $\psi \in H_{E_v}$. Using Proposition 3.7 we may choose a basis of \hat{M} and write it under the decomposition (3.1) as

$$\hat{M} \cong \prod_i \prod_{j \in \mathbb{Z}/f_i \mathbb{Z}} (R[[y_i]], \tau_{i,j}) \quad \text{with} \quad \tau_{i,j} \in R[[y_i]][\frac{1}{z-\zeta}]^\times.$$

Let $c \in \mathbb{N}_0$ be such that $(z-\zeta)^c \tau_{ij}, (z-\zeta)^c \tau_{ij}^{-1} \in R[[y_i]]$. Since the $y_i - \varphi(y_i)$ for $\varphi \in H_{E_v}$ with $(i,j)(\varphi) = (i,j)$ are prime elements in the factorial ring $R[[y_i]]$, Lemma 3.5 applied to $(z-\zeta)^c \tau_{ij} \cdot (z-\zeta)^c \tau_{ij}^{-1} = (z-\zeta)^{2c}$ shows that

$$\tau_{i,j} = \varepsilon_{i,j} \cdot \prod_{\varphi \in H_{E_v}: (i,j)(\varphi) = (i,j)} (y_i - \varphi(y_i))^{d_\varphi} \quad (4.1)$$

for a unit $\varepsilon_{i,j} \in R[[y_i]]^\times$ and integers $d_\varphi \in \mathbb{Z}$. By Definition 3.10 the tuple $\Phi = (d_\varphi)_\varphi$ is the local CM-type of \hat{M} .

We will now treat all factors of $\tau_{i,j}$ separately. We may do this because we can view \hat{M} as the tensor product $\hat{M}_{E_v,0} \otimes \bigotimes_\varphi \hat{M}_{E_v,\varphi}^{\otimes d_\varphi}$ over $\mathcal{O}_{E_v,R}$ of $\hat{M}_{E_v,0} := (\mathcal{O}_{E_v,R}, \tau_0 = \prod_{i,j} \varepsilon_{i,j})$ and all the d_φ -th powers of $\hat{M}_{E_v,\varphi} := (\mathcal{O}_{E_v,R}, \prod_{i,j} \tau_{\varphi,i,j})$ where

$$\tau_{\varphi,i,j} = \begin{cases} 1 & \text{if } (i,j) \neq (i,j)(\varphi), \\ y_i - \varphi(y_i) & \text{if } (i,j) = (i,j)(\varphi). \end{cases}$$

4.2. We first treat the case of $\hat{M}_{E_v,0} = (\mathcal{O}_{E_v,R}, \tau_0 = (\varepsilon_{i,j})_{i,j})$, where $\varepsilon_{i,j} \in R[[y_i]]^\times$. We compute the τ -invariants as $(c_{i,j})_{i,j}$ with $c_{i,j} := \sum_{n=0}^\infty c_{i,j,n} y_i^n$ subject to the condition

$$(c_{i,j})_{i,j} = \tau_0 \circ \hat{\sigma}((c_{i,j})_{i,j}), \quad \text{that is} \quad c_{i,j} = \varepsilon_{i,j} \cdot \hat{\sigma}(c_{i,j-1}) \quad \text{for all } i,j.$$

The latter implies $c_{i,j} = \varepsilon_{i,j} \cdot \hat{\sigma}(\varepsilon_{i,j-1}) \cdot \dots \cdot \hat{\sigma}^{j-1}(\varepsilon_{i,1}) \cdot \hat{\sigma}^j(c_{i,0})$ and $c_{i,0} = \varepsilon_i \cdot \hat{\sigma}^{f_i}(c_{i,0})$, where we set $\varepsilon_i := \varepsilon_{i,0} \cdot \hat{\sigma}(\varepsilon_{i,f_i-1}) \cdot \dots \cdot \hat{\sigma}^{f_i-1}(\varepsilon_{i,1}) = \sum_{n=0}^\infty b_{i,n} y_i^n \in R[[y_i]]^\times$. In particular $b_{i,0} \in R^\times$. The resulting formulas for the coefficients

$$c_{i,0,0} = b_{i,0} \cdot c_{i,0,0}^{\tilde{q}_i} \quad \text{and} \quad c_{i,0,n} - b_{i,0} \cdot c_{i,0,n}^{\tilde{q}_i} = \sum_{\ell=1}^n b_{i,\ell} \cdot c_{i,0,n-\ell}^{\tilde{q}_i},$$

where $\tilde{q}_i = q_v^{f_i}$, lead to the formulas

$$\tilde{q}_{i,0,0}^{-1} = b_{i,0}^{-1} \quad \text{and} \quad \frac{c_{i,0,n}}{c_{i,0,0}} - \left(\frac{c_{i,0,n}}{c_{i,0,0}} \right)^{\tilde{q}_i} = \sum_{\ell=1}^n \frac{b_{i,\ell}}{b_{i,0}} \cdot \left(\frac{c_{i,0,n-\ell}}{c_{i,0,0}} \right)^{\tilde{q}_i},$$

which have solutions $c_{i,0,n} \in \mathcal{O}_{L^{\text{sep}}}$ with $c_{i,0,0} \in \mathcal{O}_{L^{\text{sep}}}^\times$. In particular, the field extension of L generated by the $c_{i,j,n}$ is unramified. Then $(c_{i,j})_{i,j}$ is an \mathcal{O}_{E_v} -basis of $H_v^1(\underline{M}_0, A_v)$. Under the period isomorphism $h_{v,\text{dR}}$ it is mapped to

$$(\varepsilon_{i,j}^{-1} c_{i,j})_{i,j} \in (\mathcal{O}_{E_v} \otimes_{A_v} \mathcal{O}_{\mathbb{C}_v}[[z]])^\times \subset E_v \otimes_{Q_v} \mathbb{C}_v[[z - \zeta]] = H_{\text{dR}}^1(\hat{M}_{E_v,0}, \mathbb{C}_v[[z - \zeta]]).$$

4.3. Next we compute the period isomorphism for the local shtuka $\hat{M}_{E_v,\varphi}$ from above. For an element $0 \neq \xi \in (\pi_L) \subset R$ we consider the equation

$$\hat{\sigma}^{f_i}(\ell_{y_i,\xi}^+) = (y_i - \xi) \cdot \ell_{y_i,\xi}^+ \quad \text{for} \quad \ell_{y_i,\xi}^+ := \sum_{n=0}^{\infty} \ell_n y_i^n \in L^{\text{sep}}[[y_i]]. \quad (4.2)$$

The equation can be solved by taking $\ell_n \in L^{\text{sep}}$ with $\ell_0^{\tilde{q}_i-1} = -\xi$ and $\ell_n^{\tilde{q}_i} + \xi \ell_n = \ell_{n-1}$. This implies that $|\ell_n| = |\xi|^{\tilde{q}_i^{-n}/(\tilde{q}_i-1)} < 1$ and $\ell_n \in \mathcal{O}_{L^{\text{sep}}}$. Note that this solution is not unique, but that every other solution $\tilde{\ell}_{y_i,\xi}^+$ of (4.2) is obtained by multiplying $\ell_{y_i,\xi}^+$ by an element of $\mathbb{F}_{\tilde{v}_i}[[y_i]] = \mathcal{O}_{E_v,i}$, because $\hat{\sigma}^{f_i}\left(\frac{\tilde{\ell}_{y_i,\xi}^+}{\ell_{y_i,\xi}^+}\right) = \frac{(y_i - \xi) \cdot \tilde{\ell}_{y_i,\xi}^+}{(y_i - \xi) \cdot \ell_{y_i,\xi}^+} = \frac{\tilde{\ell}_{y_i,\xi}^+}{\ell_{y_i,\xi}^+} \in L^{\text{sep}}[[y_i]]$ is invariant under $\hat{\sigma}^{f_i}$ and hence lies in $\mathbb{F}_{\tilde{v}_i}[[y_i]]$.

According to the decomposition (3.1) the τ -invariants of $\hat{M}_{E_v,\varphi}$ have the form $\tilde{u} = (\tilde{u}_{i,j})_{i,j} \in \prod_{i,j} L^{\text{sep}}[[y_i]]$ with $\tilde{u} = \tau_\varphi \cdot \hat{\sigma}(\tilde{u})$, that is

$$\tilde{u}_{i,j} = \begin{cases} \hat{\sigma}(\tilde{u}_{i,j-1}) & \text{if } (i,j) \neq (i,j)(\varphi), \\ (y_i - \varphi(y_i)) \cdot \hat{\sigma}(\tilde{u}_{i,j-1}) & \text{if } (i,j) = (i,j)(\varphi). \end{cases}$$

For $j, j' \in \mathbb{Z}/f_i\mathbb{Z}$ we denote by (j, j') the representative of $j - j'$ in $\{0, \dots, f_i - 1\}$. This implies that $\tilde{u}_{(i,j)(\varphi)} = (y_i(\varphi) - \varphi(y_i(\varphi))) \cdot \hat{\sigma}^{f_i(\varphi)}(\tilde{u}_{(i,j)(\varphi)})$, and $\tilde{u}_{i(\varphi),j} = \hat{\sigma}^{(j,j(\varphi))}(\tilde{u}_{(i,j)(\varphi)})$, as well as $\tilde{u}_{i,j} \in \mathbb{F}_{\tilde{v}_i}[[y_i]]$ for all $i \neq i(\varphi)$ and all j . In particular an \mathcal{O}_{E_v} -basis of $H_v^1(\hat{M}_{E_v,\varphi}, A_v)$ is given by

$$\tilde{u} = (\tilde{u}_{i,j})_{i,j} \quad \text{with} \quad \tilde{u}_{i,j} = \hat{\sigma}^{(j,j(\varphi))}(\ell_{y_i,\varphi(y_i)}^+)^{-\delta_{i,i(\varphi)}} = (\ell_{y_i,\varphi(y_i)}^+)^{-\delta_{i,i(\varphi)}} \quad (4.3)$$

where $\delta_{i,i(\varphi)}$ is the Kronecker δ . The comparison isomorphism $h_{v,\text{dR}}$ sends this \tilde{u} to the element

$$\tau_\varphi^{-1} \cdot \tilde{u} = \left(\left((y_i - \varphi(y_i))^{\delta_{j,j(\varphi)}} \cdot \hat{\sigma}^{(j,j(\varphi))}(\ell_{y_i,\varphi(y_i)}^+) \right)^{-\delta_{i,i(\varphi)}} \right)_{i,j} \quad (4.4)$$

of $E_v \otimes_{Q_v} \mathbb{C}_v((z - \zeta)) = H_{\text{dR}}^1(\hat{M}_{E_v,\varphi}, \mathbb{C}_v((z - \zeta)))$.

4.4. Putting everything together we see that our $\hat{M} \cong (\mathcal{O}_{E_v,R}, \prod_{i,j} \tau_{i,j})$ with $\tau_{i,j}$ from (4.1) has

$$\tilde{u} = (\tilde{u}_{i,j})_{i,j} = \left(c_{i,j} \cdot \prod_{\varphi \in H_{E_v,i}} \hat{\sigma}^{(j,j(\varphi))}(\ell_{y_i,\varphi(y_i)}^+)^{-d_\varphi} \right)_{i,j}$$

as an \mathcal{O}_{E_v} -basis of $H_v^1(\hat{M}, A_v) \cong H_v^1(\hat{M}_{E_v,0}, A_v) \otimes \bigotimes_{\varphi \in H_{E_v}} H_v^1(\hat{M}_{E_v,\varphi}, A_v)^{\otimes d_\varphi}$, where the tensor product is over \mathcal{O}_{E_v} . Under $h_{v,\text{dR}}$ this \tilde{u} is mapped to the element

$$\tau_{\hat{M}}^{-1} \cdot \tilde{u} = \left(\varepsilon_{i,j}^{-1} c_{i,j} \cdot \prod_{\varphi \in H_{E_v,i}} \left((y_i - \varphi(y_i))^{\delta_{j,j(\varphi)}} \cdot \hat{\sigma}^{(j,j(\varphi))}(\ell_{y_i,\varphi(y_i)}^+) \right)^{-d_\varphi} \right)_{i,j} \quad (4.5)$$

of $E_v \otimes_{Q_v} \mathbb{C}_v((z - \zeta)) \cong H_{\text{dR}}^1(\hat{M}, \mathbb{C}_v((z - \zeta)))$. Note that also the de Rham cohomology is the tensor product $H_{\text{dR}}^1(\hat{M}, L[[z - \zeta]]) \cong H_{\text{dR}}^1(\hat{M}_{E_v,0}, L[[z - \zeta]]) \otimes \bigotimes_{\varphi \in H_{E_v}} H_{\text{dR}}^1(\hat{M}_{E_v,\varphi}, L[[z - \zeta]])^{\otimes d_\varphi}$ over $E_v \otimes_{Q_v} L[[z - \zeta]]$.

Remark 4.5. Note that $\underline{\hat{M}}_{E_v, \varphi} \otimes_{\mathcal{O}_{E_v}} \mathcal{O}_{E_v, i}$ with $i = i(\varphi)$ is the local $\hat{\sigma}$ -shtuka associated with a Lubin-Tate formal group, and so our treatment is analogous to Colmez's [Col93, §I.2]. Namely, let $\hat{G} = \hat{G}_{a, R} = \text{Spf } R[[X]]$ be the formal additive group over R with an action of $\mathcal{O}_{E_v, i} = \mathbb{F}_{\tilde{v}_i}[[y_i]]$ given by

$$\begin{aligned} [\lambda] : X &\longmapsto \varphi(\lambda) \cdot X = \lambda^{q_v^{j(\varphi)}} \cdot X & \text{for } \lambda \in \mathbb{F}_{\tilde{v}_i}, \\ [y_i] : X &\longmapsto X^{\tilde{q}_i} + \varphi(y_i) \cdot X. \end{aligned}$$

Then \hat{G} is the Lubin-Tate formal group over R associated with $\mathcal{O}_{\varphi(E_v, i)}$; see [LT65]. It is a z -divisible local Anderson module in the sense of [HS15, Definition 6.1]. For an element $a \in \mathcal{O}_{E_v, i}$ let $\hat{G}[a] := \ker[a]$. Under the anti-equivalence between z -divisible local Anderson modules and effective local $\hat{\sigma}$ -shtukas over $S = \text{Spec } R$ from [HS15, Theorem 7.3] the associated local shtuka is

$$\hat{M} := \hat{M}(\hat{G}) := \varprojlim_n \text{Hom}_R(\hat{G}[z^n], \mathbb{G}_{a, R}) = \varprojlim_n \text{Hom}_R(\hat{G}[y_i^{n e_i}], \mathbb{G}_{a, R}) = \bigoplus_{k=0}^{f_i-1} R[[y_i]] \tau^k$$

with $\tau^0 := \text{id} : \hat{G} \xrightarrow{\sim} \hat{G}_{a, R}$ and $\tau^k := \text{Frob}_{q_v, \hat{G}_{a, R}} \circ \tau^0 : X \mapsto X^{q_v^k}$. It is an $\mathcal{O}_{E_v, i, R}$ -module via the $\mathcal{O}_{E_v, i}$ -action on $\hat{G}[z^n]$ and the R -action on $\mathbb{G}_{a, R}$, and is equipped with the Frobenius $\tau_{\hat{M}} : \hat{\sigma}^* \hat{M} \rightarrow \hat{M}$ given by $\hat{\sigma}_M^* m \mapsto \text{Frob}_{q_v, \mathbb{G}_{a, R}} \circ m$ for $m \in \hat{M}$. We set $\underline{\hat{M}}(\hat{G}) := (\hat{M}, \tau_{\hat{M}})$. In particular, we see that $\lambda \in \mathbb{F}_{\tilde{v}_i}$ acts on $R[[y_i]] \tau^k$ as $\lambda^{q_v^{k+j(\varphi)}}$ and so $\hat{M}/\mathfrak{b}_{i, j} \hat{M} = R[[y_i]] \tau^{(j, j(\varphi))}$ under the decomposition (3.1). Since $\tau_{\hat{M}}(\hat{\sigma}_M^* \tau^{f_i-1}) = \tau^{f_i} = [y_i] - \varphi(y_i) : X \mapsto X^{\tilde{q}_i} = ([y_i] - \varphi(y_i))(X)$, we see that

$$\tau_{\hat{M}} = (\tau_{\hat{M}, j})_j \quad \text{with} \quad \tau_{\hat{M}, j} = \begin{cases} 1 & \text{if } j \neq j(\varphi), \\ y_i - \varphi(y_i) & \text{if } j = j(\varphi), \end{cases}$$

that is, $\underline{\hat{M}}(\hat{G}) = \underline{\hat{M}}_{E_v, \varphi} \otimes_{\mathcal{O}_{E_v}} \mathcal{O}_{E_v, i}$.

Moreover, if we want to also consider the other components of $\underline{\hat{M}}_{E_v, \varphi}$ for $i \neq i(\varphi)$ we take the divisible local Anderson module $\hat{G}_{E_v, \varphi} := \hat{G} \times \prod_{i \neq i(\varphi)} (E_{v, i} / \mathcal{O}_{E_v, i})_R$. It has local shtuka $\underline{\hat{M}}(\hat{G}_{E_v, \varphi}) = \underline{\hat{M}}(\hat{G}) \oplus \bigoplus_{i \neq i(\varphi)} \underline{\hat{M}}((E_{v, i} / \mathcal{O}_{E_v, i})_R) = \underline{\hat{M}}_{E_v, \varphi}$, because $\underline{\hat{M}}((E_{v, i} / \mathcal{O}_{E_v, i})_R) = (\mathcal{O}_{E_v, i, R}, \tau = 1)$.

4.6. We want to describe the Galois action of \mathcal{G}_L on $H_v^1(\underline{\hat{M}}_{E_v, \varphi}, A_v)$. Recall from [HK15, Definition 4.7, Proposition 4.8 and Remark 4.9] that the Tate module of \hat{G} is defined as $T_v \hat{G} := \text{Hom}_{A_v}(Q_v / A_v, \hat{G}(L^{\text{sep}}))$ and that there is a perfect pairing of A_v -modules

$$T_v \hat{G} \times H_v^1(\underline{\hat{M}}(\hat{G}), A_v) \longrightarrow \text{Hom}_{\mathbb{F}_v}(Q_v / A_v, \mathbb{F}_v), \quad (f, m) \longmapsto m \circ f, \quad (4.6)$$

which is equivariant for the actions of \mathcal{G}_L and $\text{End}_R(\underline{\hat{M}}(\hat{G})) = \text{End}_R(\hat{G})^{\text{op}}$. Here the A_v -module $\text{Hom}_{\mathbb{F}_v}(Q_v / A_v, \mathbb{F}_v) \cong \hat{\Omega}_{A_v / \mathbb{F}_v}^1 \cong \mathbb{F}_v[[z]] dz$ is free of rank one; see [HK15, before Proposition 4.8]. We have already computed $H_v^1(\underline{\hat{M}}_{E_v, \varphi}, A_v) = E_{v, i} \cdot (\check{u}_{i, j})_{i, j}$ in (4.3). We will now compute $T_v \hat{G}_{E_v, \varphi}$ and the action of \mathcal{G}_L on both $T_v \hat{G}_{E_v, \varphi}$ and $H_v^1(\underline{\hat{M}}_{E_v, \varphi}, A_v)$. Let again $i = i(\varphi)$. Since $\mathcal{O}_{E_v, i}$ acts on $\hat{G}(L^{\text{sep}})$ we have

$$\begin{aligned} T_v \hat{G} &= \text{Hom}_{\mathcal{O}_{E_v, i}}(\mathcal{O}_{E_v, i} \otimes_{A_v} (Q_v / A_v), \hat{G}(L^{\text{sep}})) \\ &= \text{Hom}_{\mathcal{O}_{E_v, i}}(E_{v, i} / \mathcal{O}_{E_v, i}, \hat{G}(L^{\text{sep}})) && \ni f \\ &= \{(P_n)_n \in \prod_{n \in \mathbb{N}_0} \hat{G}[y_i^n](L^{\text{sep}}) : [y_i](P_n) = P_{n-1}\} && \ni (P_n)_n := (f(y_i^{-n}))_n, \end{aligned}$$

where f is reconstructed from $(P_n)_n$ as $f(ay_i^{-n}) := [a](P_n)$ for $a \in \mathcal{O}_{E_{v,i}}^\times$. From equation (4.2) we see that $\ell_{y_i, \varphi(y_i)}^+ = \sum_{n=0}^\infty \ell_n y_i^n$ satisfies

$$[y_i](\ell_0) = \ell_0^{\tilde{q}_i} + \varphi(y_i)\ell_0 = 0 \quad \text{and} \quad [y_i](\ell_n) = \ell_n^{\tilde{q}_i} + \varphi(y_i)\ell_n = \ell_{n-1}.$$

Thus $\ell_{n-1} \in \hat{G}[y_i^n](L^{\text{sep}})$ and $T_v \hat{G} = \mathcal{O}_{E_{v,i}} \cdot (\ell_{n-1})_n$. To compute the \mathcal{G}_L -action on $T_v \hat{G}$ we need the following

Proposition 4.7. *Let $\mathbb{F}_{\tilde{v}_i}((\varphi(y_i)))_\infty := \mathbb{F}_{\tilde{v}_i}((\varphi(y_i)))(\ell_n : n \in \mathbb{N}_0)$. Then there is an isomorphism of topological groups*

$$\chi : \text{Gal}(\mathbb{F}_{\tilde{v}_i}((\varphi(y_i)))_\infty / \mathbb{F}_{\tilde{v}_i}((\varphi(y_i)))) \xrightarrow{\sim} \mathbb{F}_{\tilde{v}_i}[[y_i]]^\times = \mathcal{O}_{E_{v,i}}^\times$$

satisfying $g(\ell_{y_i, \varphi(y_i)}^+) := \sum_{n=0}^\infty g(\ell_n) y_i^n = \chi(g) \cdot \ell_{y_i, \varphi(y_i)}^+$ in $\mathbb{F}_{\tilde{v}_i}((\varphi(y_i)))_\infty[[y_i]]$ for g in the Galois group. The isomorphism χ is independent of the choice of $\ell_{y_i, \varphi(y_i)}^+$.

Proof. The existence of χ follows from the equation $\hat{\sigma}^{f_i}(\ell_{y_i, \varphi(y_i)}^+) = (y_i - \varphi(y_i)) \cdot \ell_{y_i, \varphi(y_i)}^+$, which implies that $\chi(g) := \frac{g(\ell_{y_i, \varphi(y_i)}^+)}{\ell_{y_i, \varphi(y_i)}^+}$ is $\hat{\sigma}^{f_i}$ -invariant, that is $\chi(g) \in \mathbb{F}_{\tilde{v}_i}[[y_i]]^\times$. Furthermore, χ is an isomorphism because ℓ_{n-1} is a uniformizing parameter of $\mathbb{F}_{\tilde{v}_i}((\varphi(y_i)))(\ell_0, \dots, \ell_{n-1})$ and so the equations defining the ℓ_n are irreducible by Eisenstein. Every other solution of (4.2) is of the form $a \cdot \ell_{y_i, \varphi(y_i)}^+$ with $a \in \mathbb{F}_{\tilde{v}_i}[[y_i]]$ and so $g(a \cdot \ell_{y_i, \varphi(y_i)}^+) = a \cdot g(\ell_{y_i, \varphi(y_i)}^+) = \chi(g) \cdot a \cdot \ell_{y_i, \varphi(y_i)}^+$. This shows that $\chi(g)$ does not depend on the solution $\ell_{y_i, \varphi(y_i)}^+$. \square

Let $\mathcal{I}_L \subset \mathcal{G}_L$ be the inertia subgroup and similarly for other fields. By local class field theory, see Lubin and Tate [LT65, Corollary on p. 386], the image of $g \in \mathcal{I}_{\varphi(E_{v,i})}$ in $\mathcal{G}_{\varphi(E_{v,i})}^{\text{ab}}$ equals the norm residue symbol $(\chi(g)^{-1}|_{y_i=\varphi(y_i)}, \varphi(E_{v,i})^{\text{ab}}/\varphi(E_{v,i}))$ where $\varphi(E_{v,i})^{\text{ab}}$ is the maximal abelian extension of $\varphi(E_{v,i})$ in Q_v^{sep} . In general, the homomorphism $\chi_L : \mathcal{I}_L \rightarrow \mathcal{O}_L^\times$ with $g|_{L^{\text{ab}}} = (\chi_L(g)^{-1}, L^{\text{ab}}/L)$ is sometimes called the *character of local class field theory* of the field L . So we see that $\chi(g)|_{y_i=\varphi(y_i)} = \chi_{\varphi(E_{v,i})}(g)$. If L is separable over $\varphi(E_{v,i})$ these characters are compatible for $g \in \mathcal{I}_L$ in the sense that $\chi_{\varphi(E_{v,i})}(g) = N_{L/\varphi(E_{v,i})}(\chi_L(g))$.

4.8. From $T_v \hat{G} = \mathcal{O}_{E_{v,i}} \cdot (\ell_{n-1})_n$ it follows that g acts on $T_v \hat{G}$ in the same way as an endomorphism in $\mathcal{O}_{E_{v,i}}^\times$. Let us compute this endomorphism. We write $\chi(g) = \sum_{k=0}^\infty a_k y_i^k$ with $a_k \in \mathbb{F}_{\tilde{v}_i}$. Then the expansion $g(\ell_{y_i, \varphi(y_i)}^+) = \chi(g) \cdot \ell_{y_i, \varphi(y_i)}^+ = \sum_{n=0}^\infty \sum_{k=0}^n a_k \ell_{n-k} y_i^n$ implies that $g(\ell_n) = \sum_{k=0}^n a_k \ell_{n-k} = \sum_{k=0}^n \varphi(a_k^{q_v^{-j(\varphi)}})[y_i^k](\ell_n)$. Thus every element $g \in \mathcal{I}_L$ acts on $T_v \hat{G}$ as the endomorphism $\sum_{k=0}^\infty a_k^{q_v^{-j(\varphi)}} y_i^k = \hat{\sigma}^{-j(\varphi)}(\chi(g)) = \varphi^{-1}(\chi(g)|_{y_i=\varphi(y_i)}) = \varphi^{-1} \circ \chi_{\varphi(E_{v,i})}(g) \in \mathcal{O}_{E_{v,i}}^\times$ and on $T_v \hat{G}_{E_{v,\varphi}}$ as the endomorphism $(\hat{\sigma}^{-j(\varphi)}(\chi(g))^{\delta_{i,i(\varphi)}})_i \in \mathcal{O}_{E_v}^\times$.

Definition 4.9. We define the character $\chi_{E_{v,\varphi}} := ((\varphi^{-1} \circ \chi_{\varphi(E_{v,i})})^{\delta_{i,i(\varphi)}})_i : \mathcal{I}_L \rightarrow \mathcal{O}_{E_v}^\times$ by mapping $g \mapsto (\hat{\sigma}^{-j(\varphi)}(\chi(g))^{\delta_{i,i(\varphi)}})_i = (1, \dots, 1, \varphi^{-1} \circ \chi_{\varphi(E_{v,i})}(g), 1, \dots, 1)$.

4.10. Due to the equivariance of the pairing (4.6) under \mathcal{G}_L and $\text{End}_R(\hat{G}_{E_{v,\varphi}})$ the action of $g \in \mathcal{G}_L$ on $H_v^1(\hat{M}(\hat{G}_{E_{v,\varphi}}, A_v))$ is given by the endomorphism $\chi_{E_{v,\varphi}}(g)^{-1}$. We can also compute this action directly as follows. It factors through the restriction of g to $\text{Gal}(\mathbb{F}_{\tilde{v}_i}((\varphi(y_i)))_\infty / \mathbb{F}_{\tilde{v}_i}((\varphi(y_i))))$ which we denote

again by g . Then on the basis $(\check{u}_{i,j})_j$ of $H_v^1(\hat{M}(\hat{G}), A_v)$ from (4.3) we compute

$$\begin{aligned}
g(\check{u}_{i,j})_j &= g\left(\hat{\sigma}^{(j,j(\varphi))}(\ell_{y_i,\varphi(y_i)}^+)^{-\delta_{i,i(\varphi)}}\right)_j \\
&= \left(\hat{\sigma}^{(j,j(\varphi))}(\chi(g) \cdot \ell_{y_i,\varphi(y_i)}^+)^{-\delta_{i,i(\varphi)}}\right)_j \\
&= \left(\hat{\sigma}^{j-j(\varphi)}(\chi(g)^{-1}) \cdot \hat{\sigma}^{(j,j(\varphi))}(\ell_{y_i,\varphi(y_i)}^+)^{-1}\right)_j \\
&= (\hat{\sigma}^{-j(\varphi)}(\chi(g)^{-1}) \otimes 1) \cdot (\check{u}_{i,j})_j
\end{aligned}$$

for the element $\hat{\sigma}^{-j(\varphi)}(\chi(g)^{-1}) \otimes 1 \in \mathcal{O}_{E_{v,i}} \otimes_{A_v} R[[z]]$. That is, the action of $g \in \mathcal{G}_L$ on $H_v^1(\hat{M}(\hat{G}), A_v)$ coincides with the endomorphism $\hat{\sigma}^{-j(\varphi)}(\chi(g)^{-1})$ and the action of $g \in \mathcal{I}_L$ on $H_v^1(\hat{M}(\hat{G}_{E_v,\varphi}), A_v) = H_v^1(\hat{M}_{E_v,\varphi}, A_v)$ coincides with the endomorphism $\chi_{E_v,\varphi}(g)^{-1} \in \mathcal{O}_{E_v}^\times$.

Proposition 4.11. *Let \hat{M} have complex multiplication by a commutative, semi-simple Q_v -algebra E_v with CM-type $\Phi = (d_\varphi)_{\varphi \in H_{E_v}}$. Then the action of $g \in \mathcal{I}_L$ on $H_v^1(\hat{M}, A_v)$ coincides with the endomorphism $\prod_{\varphi \in H_{E_v}} \chi_{E_v,\varphi}(g)^{-d_\varphi} \in \mathcal{O}_{E_v}^\times$.*

Proof. This follows from the computations in 4.4, 4.10 and 4.2 by observing that \mathcal{I}_L acts trivially on $H_v^1(\hat{M}_{E_v,0}, A_v)$, because its generator $(c_{i,j})_{i,j}$ is defined over the maximal unramified extension of L . \square

4.12. To compute the absolute value $|\int_u \omega|_v$ we again treat each factor $\hat{M}_{E_v,0}$ and $\hat{M}_{E_v,\varphi}$ of \hat{M} separately. We begin with $\hat{M}_{E_v,\varphi}$ and set $i := i(\varphi)$. Let $\omega_\psi^\circ := 1 \in H^\psi(\hat{M}_{E_v,\varphi}, L[[y_i - \psi(y_i)]])$. It is a generator as $L[[y_i - \psi(y_i)]]$ -module as in Definition 3.9, and is mapped under the period isomorphism of $\hat{M}_{E_v,\varphi}$ from (4.4) to

$$h_{v,\text{dR}}^{-1}(\omega_\psi^\circ) = (0, \dots, \left((y_{i(\psi)} - \varphi(y_{i(\psi)}))^{\delta_{j(\psi),j(\varphi)}} \cdot \hat{\sigma}^{(j(\psi),j(\varphi))}(\ell_{y_{i(\psi)},\varphi(y_{i(\psi)})}^+)\right)^{\delta_{i(\psi),i(\varphi)}}, \dots, 0) \cdot \check{u}, \quad (4.7)$$

where the non-zero entry is in component ψ . We denote this entry by $\Omega(E_v, \varphi, \psi)$. It is analogous to Colmez's [Col93, Théorème I.2.1] element of \mathbf{B}_{dR} with the same name which satisfies the following

Theorem 4.13. *Let $\varphi, \psi \in H_{E_v}$ satisfy $i(\varphi) = i(\psi) =: i$ and assume that $E_{v,i}$ is separable over Q_v . Then the element*

$$\Omega(E_v, \varphi, \psi) := (y_i - \varphi(y_i))^{\delta_{j(\psi),j(\varphi)}} \cdot \hat{\sigma}^{(j(\psi),j(\varphi))}(\ell_{y_i,\varphi(y_i)}^+) \in \mathbb{C}_v((y_i - \psi(y_i))) = \mathbb{C}_v((z - \zeta))$$

satisfies

(a) $\hat{v}(\Omega(E_v, \varphi, \psi)) = 1$ if $\varphi = \psi$ and $\hat{v}(\Omega(E_v, \varphi, \psi)) = 0$ if $\varphi \neq \psi$.

(b)

$$v(\Omega(E_v, \varphi, \psi)) = \begin{cases} \frac{1}{e_i(\bar{q}_i - 1)} - v(\mathfrak{D}_{\varphi(E_{v,i})/Q_v}) & \text{if } \varphi = \psi, \\ \frac{1}{e_i(\bar{q}_i - 1)} + v(\psi(y_i) - \varphi(y_i)) & \text{if } \varphi \neq \psi \text{ and } j(\varphi) = j(\psi), \\ \frac{q_v^{(j(\psi),j(\varphi))}}{e_i(\bar{q}_i - 1)} & \text{if } j(\varphi) \neq j(\psi), \end{cases}$$

where $\mathfrak{D}_{\varphi(E_{v,i})/Q_v}$ is the different of $\varphi(E_{v,i})$ over Q_v .

(c) If $g \in \mathcal{I}_L$, then $g(\Omega(E_v, \varphi, \psi)) = \psi(\chi_{E_v,\varphi}(g)) \cdot \Omega(E_v, \varphi, \psi)$. Note that if L is separable over Q_v then $\psi(\chi_{E_v,\varphi}(g)) = \psi(\varphi^{-1}(N_{L/\varphi(E_{v,i})}\chi_L(g)))$.

(d) Let $u \in H_{1,v}(\hat{M}_{E_v,\varphi}, A_v) := \text{Hom}_{A_v}(H_{1,v}(\hat{M}_{E_v,\varphi}, A_v), A_v)$ be a generator as \mathcal{O}_{E_v} -module and let ω_ψ° be an $L[[y_i - \psi(y_i)]]$ -generator of $H^\psi(\hat{M}_{E_v,\varphi}, L[[y_i - \psi(y_i)]])$ subject to the conditions in Definition 3.9, that is subject to $\omega_\psi^\circ \bmod y_i - \varphi(y_i) \in H_v^1(\hat{M}_{E_v,\varphi}, L)$ being an R -generator of $H_v^1(\hat{M}_{E_v,\varphi}, R)$. Then $\int_u \omega_\psi^\circ := u \otimes \text{id}_{\mathbb{C}_v((z-\zeta))}(h_{v,\text{dR}}^{-1}(\omega_\psi^\circ)) \in \mathbb{C}_v((z-\zeta))$ equals $\Omega(E_v, \varphi, \psi)$ up to multiplication by an element of $R^\times + (z-\zeta) \cdot L[[z-\zeta]]$.

Remark 4.14. Note that in contrast to the number field case [Col93, Théorème I.2.1] the element $\Omega(E_v, \varphi, \psi) \in \mathbb{C}_v((z-\zeta))$ is by (a), (b) and (c) uniquely determined only up to multiplication by an element of $\mathcal{O}_L^\times + (z-\zeta) \cdot \tilde{L}[[z-\zeta]]$, where \tilde{L} is the completion of the compositum of $\mathbb{F}_q^{\text{alg}}$ with the perfect closure of L in Q_v^{alg} , because the fixed field of \mathcal{I}_L in $\mathbb{C}_v((z-\zeta))$ equals $\tilde{L}((z-\zeta))$ by the Ax-Sen-Tate theorem [Ax70].

Proof of Theorem 4.13. In 4.3 we have seen that the coefficients of the series $\ell_{y_i, \varphi(y_i)}^+ = \sum_{n=0}^\infty \ell_n y_i^n$ satisfy $v(\ell_n) = v(\varphi(y_i)) \cdot \tilde{q}_i^{-n}/(\tilde{q}_i - 1)$. From $v(\psi(y_i)) = 1/e_i = v(\varphi(y_i))$ it follows that the evaluation of $\hat{\sigma}^{(j(\psi), j(\varphi))}(\ell_{y_i, \varphi(y_i)}^+) \big|_{y_i=\psi(y_i)} = \sum_{n=0}^\infty \ell_n^{(j(\psi), j(\varphi))} \psi(y_i)^n$ at $y_i = \psi(y_i)$ satisfies

$$v(\ell_n^{(j(\psi), j(\varphi))} \psi(y_i)^n) = \frac{1}{e_i} \cdot \left(n + \frac{q_v^{(j(\psi), j(\varphi))}}{\tilde{q}_i^n (\tilde{q}_i - 1)} \right). \quad (4.8)$$

Since $0 \leq (j(\psi), j(\varphi)) \leq f_i - 1$ the second fraction in the parenthesis is strictly smaller than 1, and so the valuations in (4.8) are strictly increasing with n and attain their minimum $\frac{q_v^{(j(\psi), j(\varphi))}}{e_i (\tilde{q}_i - 1)}$ for $n = 0$. This shows that $\hat{\sigma}^{(j(\psi), j(\varphi))}(\ell_{y_i, \varphi(y_i)}^+) \big|_{y_i=\psi(y_i)}$ is non-zero in L and

$$v\left(\hat{\sigma}^{(j(\psi), j(\varphi))}(\ell_{y_i, \varphi(y_i)}^+) \big|_{y_i=\psi(y_i)}\right) = \frac{q_v^{(j(\psi), j(\varphi))}}{e_i (\tilde{q}_i - 1)}.$$

In particular the valuation $\hat{v}(\hat{\sigma}^{(j(\psi), j(\varphi))}(\ell_{y_i, \varphi(y_i)}^+)) = 0$.

(a) Lemma A.1 implies that in the ψ -component of $E_v \otimes_{Q_v} L[[z-\zeta]]$ we have $\text{ord}_{y_i - \psi(y_i)} = \text{ord}_{z-\zeta}$. If $j(\varphi) = j(\psi)$, that is $\varphi|_{\mathbb{F}_{\tilde{v}_i}} = \psi|_{\mathbb{F}_{\tilde{v}_i}}$ then $\varphi \neq \psi$ implies $\psi(y_i) - \varphi(y_i) \neq 0$ in L , because $E_{v,i} = \mathbb{F}_{\tilde{v}_i}((y_i))$. Therefore the valuation \hat{v} of $y_i - \varphi(y_i) = (\psi(y_i) - \varphi(y_i)) + (y_i - \psi(y_i))$ equals zero for $\varphi \neq \psi$ and $j(\varphi) = j(\psi)$. This implies (a).

(b) We will calculate $v((\Omega(E_v, \varphi, \psi)))$ in three different cases separately as follows.

Case1: $\psi = \varphi$. In this case $\hat{v}(\Omega(E_v, \varphi, \psi)) = 1$ and so

$$\begin{aligned} v(\Omega(E_v, \varphi, \psi)) &= v\left(\left(\frac{y_i - \varphi(y_i)}{z - \zeta} \cdot \ell_{y_i, \varphi(y_i)}^+\right) \big|_{y_i=\varphi(y_i)}\right) \\ &= v\left(\frac{y_i - \varphi(y_i)}{z - \zeta} \big|_{y_i=\varphi(y_i)}\right) + v(\ell_{y_i, \varphi(y_i)}^+ \big|_{y_i=\varphi(y_i)}) \\ &= -v(\mathfrak{D}_{\varphi(E_v, i)/Q_v}) + \frac{1}{e_i (\tilde{q}_i - 1)} \end{aligned}$$

by Corollary 3.6.

Case 2: $\psi \neq \varphi$ and $j(\psi) = j(\varphi)$. In this case $\hat{v}(\Omega(E_v, \varphi, \psi)) = 0$ and so

$$\begin{aligned} v(\Omega(E_v, \varphi, \psi)) &= v\left((y_i - \varphi(y_i)) \cdot \ell_{y_i, \varphi(y_i)}^+ \big|_{y_i=\psi(y_i)}\right) \\ &= v(\psi(y_i) - \varphi(y_i)) + v(\ell_{y_i, \varphi(y_i)}^+ \big|_{y_i=\psi(y_i)}) \\ &= v(\psi(y_i) - \varphi(y_i)) + \frac{1}{e_i (\tilde{q}_i - 1)}. \end{aligned}$$

Case 3: $j(\psi) \neq j(\varphi)$. In this case $\hat{v}(\Omega(E_v, \varphi, \psi)) = 0$ and so

$$v(\Omega(E_v, \varphi, \psi)) = v\left(\hat{\sigma}^{(j(\psi), j(\varphi))}(\ell_{y_i, \varphi(y_i)}^+)|_{y_i=\psi(y_i)}\right) = \frac{q_v^{(j(\psi), j(\varphi))}}{e_i(\bar{q}_i - 1)}.$$

(c) We have seen in 4.10 that $g(\tilde{u}) = \chi_{E_v, \varphi}(g)^{-1} \cdot \tilde{u}$ and $g(0, \dots, \Omega(E_v, \varphi, \psi), \dots, 0) \cdot g(\tilde{u}) = h_{v, \text{dR}}^{-1}(g(\omega_\psi^\circ)) = h_{v, \text{dR}}^{-1}(\omega_\psi^\circ) = (0, \dots, \Omega(E_v, \varphi, \psi), \dots, 0) \cdot \tilde{u}$. Thus g acts on the coefficient $(0, \dots, \Omega(E_v, \varphi, \psi), \dots, 0)$ as multiplication with $\chi_{E_v, \varphi}(g)$ and on its ψ -component $\Omega(E_v, \varphi, \psi)$ by multiplication with $\psi(\chi_{E_v, \varphi}(g))$.

(d) Let $u^\circ \in H_{1, v}(\hat{M}_{E_v, \varphi}, A_v)$ be the generator with $\langle a u^\circ, b \tilde{u} \rangle = \text{Tr}_{E_v/Q_v}(ab)$ for $a, b \in \mathcal{O}_{E_v}$, see Lemma 4.15 below. If $\omega_\psi^\circ = 1$ is the generator from 4.12 then

$$\int_u \omega_\psi^\circ = \text{Tr}_{E_v/Q_v}(0, \dots, \Omega(E_v, \varphi, \psi), \dots, 0) = \Omega(E_v, \varphi, \psi).$$

Any other generator u is of the form $u = a u^\circ$ with $a \in \mathcal{O}_{E_v}^\times$ and any other ω_ψ° differs from $\omega_\psi^\circ = 1$ by multiplication by an element $b \in R^\times + (y_i - \varphi(y_i)) \cdot L[y_i - \varphi(y_i)] \subset E_v \otimes_{Q_v} L[y_i - \varphi(y_i)]$. The product ab in $E_v \otimes_{Q_v} L[y_i - \varphi(y_i)]$ still lies in $R^\times + (y_i - \varphi(y_i)) \cdot L[y_i - \varphi(y_i)] = R^\times + (z - \zeta) \cdot L[z - \zeta]$ and under the pairing $\langle \cdot, \cdot \rangle$ it leads to $\int_u \omega_\psi^\circ = ab \cdot \Omega(E_v, \varphi, \psi)$. \square

It remains to record the following well known

Lemma 4.15. *If $E_{v,i}/Q_v$ is separable then for any perfect pairing $\langle \cdot, \cdot \rangle: \mathcal{O}_{E_{v,i}} \times \mathcal{O}_{E_{v,i}} \rightarrow A_v$ satisfying $\langle a, b \rangle = \langle ab, 1 \rangle = \langle 1, ab \rangle$, there is an element $c \in \mathcal{O}_{E_{v,i}}^\times$ with $\langle a, b \rangle = \text{Tr}_{E_{v,i}/Q_v}(abc)$.* \square

4.16. Analogously to [Col93, §I.2], we can give a uniform formula for $v(\Omega(E_v, \varphi, \psi))$ by introducing certain measures on \mathcal{G}_{Q_v} . Let $\mathcal{C}(\mathcal{G}_{Q_v}, \mathbb{Q})$ be the \mathbb{Q} -vector space of locally constant functions $a: \mathcal{G}_{Q_v} \rightarrow \mathbb{Q}$. If K is a finite separable extension of Q_v , and $\varphi, \psi \in H_K$, let $a_{K, \varphi, \psi} \in \mathcal{C}(\mathcal{G}_{Q_v}, \mathbb{Q})$ be the function given by

$$a_{K, \varphi, \psi}(g) := \begin{cases} 1 & \text{if } g\varphi = \psi, \\ 0 & \text{otherwise.} \end{cases}$$

Note that the $a_{K, \varphi, \psi}$ span $\mathcal{C}(\mathcal{G}_{Q_v}, \mathbb{Q})$.

If $L \subset Q_v^{\text{sep}}$ is a finite Galois extension of Q_v , let $\mu_L \in \mathcal{C}(\mathcal{G}_{Q_v}, \mathbb{Q})$ be the function given by the formula

$$\mu_L(g) := \begin{cases} 0 & \text{if } g \notin \mathcal{I}_{Q_v}, \\ -v(g(\pi_L) - \pi_L) & \text{if } g \in \mathcal{I}_{Q_v} \text{ and } g(\pi_L) \neq \pi_L, \\ v(\mathfrak{D}_{L/Q_v}) & \text{if } g \in \mathcal{I}_{Q_v} \text{ and } g(\pi_L) = \pi_L, \end{cases}$$

where π_L is a uniformizer of L and \mathfrak{D}_{L/Q_v} is the different of L over Q_v .

Moreover, we let e_K be the ramification index of K over Q_v and f_K the degree of the residue field of K over \mathbb{F}_v . We let $W_L^n := \{g \in \mathcal{G}_{Q_v}: g(x) \equiv x^{q_v^n} \pmod{\mathfrak{m}_{Q_v^{\text{alg}}}} \forall x \in \mathcal{O}_{Q_v^{\text{alg}}}\} / \mathcal{I}_L$. It is in bijection with $\{g \in \text{Gal}(L/Q_v): g(x) \equiv x^{q_v^n} \pmod{(\pi_L)}\}$ under the map $\mathcal{G}_{Q_v} \twoheadrightarrow \text{Gal}(L/Q_v)$.

Lemma 4.17. *Let $K, L \subset Q_v^{\text{sep}}$ be finite separable extensions of Q_v with L finite Galois over Q_v containing all the conjugates of K , and let $\varphi, \psi \in H_K$. The function $a_{K, \varphi, \psi}$ is constant modulo \mathcal{G}_L and hence can be considered as a function on $G_L := \text{Gal}(L/Q_v)$. Then*

$$\sum_{g \in G_L} a_{K, \varphi, \psi}(g) \cdot \mu_L(g) = \begin{cases} 0 & \text{if } j(\varphi) \neq j(\psi), \\ v(\mathfrak{D}_{\psi(K)/Q_v}) & \text{if } \varphi = \psi, \\ -v(\psi(\pi_K) - \varphi(\pi_K)) & \text{if } j(\varphi) = j(\psi) \text{ and } \varphi \neq \psi, \end{cases}$$

and

$$\frac{1}{e_L} \sum_{n=1}^{\infty} \sum_{g \in W_L^n} \frac{a_{K, \varphi, \psi}(g)}{q_v^{ns}} = \frac{1}{e_K} \frac{q_v^{(j(\psi), j(\varphi))s}}{q_v^{f_K s} - 1}.$$

In particular, the left hand side of both equations does not depend on the choice of L .

Proof. The proof follows in the same way as [Col93, Lemma I.2.4]. \square

Since the $a_{K,\varphi,\psi}$ generate the vector space $\mathcal{C}(\mathcal{G}_{Q_v}, \mathbb{Q})$, we get the following proposition.

Proposition 4.18. *There exist \mathbb{Q} -linear homomorphisms $Z_v(\cdot, s) : \mathcal{C}(\mathcal{G}_{Q_v}, \mathbb{Q}) \rightarrow \mathbb{C}$ if $s \in \mathbb{C}$ and $\mu_{\text{Art},v} : \mathcal{C}(\mathcal{G}_{Q_v}, \mathbb{Q}) \rightarrow \mathbb{Q}$ defined by the following formulas: if $a \in \mathcal{C}(\mathcal{G}_{Q_v}, \mathbb{Q})$ and if $L \subset Q_v^{\text{sep}}$ is a finite Galois extension of Q_v such that a is constant modulo \mathcal{G}_L , then*

$$\mu_{\text{Art},v}(a) = \sum_{g \in G_L} a(g) \cdot \mu_L(g)$$

with $G_L := \text{Gal}(L/Q_v)$, and $Z_v(a, s)$ is obtained by meromorphic extension from the following formula, valid for $\text{Re}(s) > 0$:

$$Z_v(a, s) = \frac{1}{e_L} \sum_{n=1}^{\infty} \sum_{g \in W_L^n} \frac{a(g)}{q_v^{ns}} \quad \square$$

Remark 4.19. If V is a finite dimensional \mathbb{C} -vector space, $\rho : \mathcal{G}_{Q_v} \rightarrow \text{Aut}_{\mathbb{C}}(V)$ is a continuous complex representation of \mathcal{G}_{Q_v} , and if $\chi \in \mathcal{C}^0(\mathcal{G}_{Q_v}, \mathbb{Q}) \otimes_{\mathbb{Q}} \mathbb{C}$ is the character of ρ , then $\mu_{\text{Art},v}(\chi)$ is nothing else than the degree at v of the conductor \mathfrak{f}_{χ} of χ ; cf. [Ser79, Chapter VI, § 2], where $\mu_{\text{Art},v}(\chi)$ is denoted by $f(\chi)$. And if W is the sub vector space of V stable by I_{Q_v} , we have

$$Z_v(\chi, a) \log q_v = -\frac{d}{ds} \log \left(\det(1 - q_v^{-s} \rho(\text{Frob}_{L/Q_v})|_W)^{-1} \right)$$

by [Tat84, Chapter 0, § 4] or [Ros02, Lemma 9.14]. So the linear maps $\mu_{\text{Art},v}$ and $Z_v(\cdot, s)$ coincide with the maps with the same names in Definition 1.2 in the introduction.

As a direct consequence of Theorem 4.13, Proposition 4.18 and Lemma 4.17 we get the following

Theorem 4.20. *If $\varphi, \psi \in H_{E_v}$ satisfy $i(\varphi) = i(\psi) =: i$ and $E_{v,i}$ is separable over Q_v then*

$$v(\Omega(E_v, \varphi, \psi)) = Z_v(a_{E_v, \psi, \varphi}, 1) - \mu_{\text{Art},v}(a_{E_v, \psi, \varphi}),$$

where we set $a_{E_v, \psi, \varphi} := a_{E_v, i, \psi, \varphi} : g \mapsto \delta_{g\psi, \varphi}$. \square

Definition 4.21. If $u \in H_{1,v}(\hat{M}, Q_v) := \text{Hom}_{A_v}(H_v^1(\hat{M}_{E_v, \varphi}, A_v), Q_v)$ is an E_v -generator there is an $a \in E_v^{\times}$, unique up to multiplication with an element of $\mathcal{O}_{E_v}^{\times}$, such that $a^{-1}u$ is an \mathcal{O}_{E_v} -generator of $H_{1,v}(\hat{M}, A_v)$. Then we define the valuation $v_{\psi}(u) := v(\psi(a)) \in \mathbb{Z}$.

Note that if $\underline{M} = (M, \tau_M)$ is a uniformizable A -motive over L with good model \underline{M} and $\hat{M} = \hat{M}_v(\underline{M})$ is the local shtuka at v associated with \underline{M} as in Example 2.2, then for an E -generator $u \in H_{1, \text{Betti}}(\underline{M}, Q)$ the present definition of $v_{\psi}(h_{\text{Betti},v}(u))$ coincides with the definition of $v_{\psi}(u)$ from (1.12).

Corollary 4.22. *Let $\varphi, \psi \in H_{E_v}$ with $i(\varphi) = i(\psi) =: i$ and assume that $E_{v,i}$ is separable over Q_v . Let $u \in H_{1,v}(\hat{M}_{E_v, \varphi}, Q_v)$ be a generator as E_v -module and let ω_{ψ} be an $L[[y_i - \psi(y_i)]]$ -generator of $H^{\psi}(\hat{M}_{E_v, \varphi}, L[[y_i - \psi(y_i)]])$. Then $\int_u \omega_{\psi} := u \otimes \text{id}_{\mathbb{C}_v((z-\zeta))}(h_{v, \text{dR}}^{-1}(\omega_{\psi}))$ has valuation*

$$v(\int_u \omega_{\psi}) = Z_v(a_{E_v, \psi, \varphi}, 1) - \mu_{\text{Art},v}(a_{E_v, \psi, \varphi}) + v(\omega_{\psi}) + v_{\psi}(u),$$

where $v(\omega_{\psi})$ and $v_{\psi}(u)$ were defined in Definitions 3.9 and 4.21.

Proof. Let $a \in E_v$ be such that $u^\circ := a^{-1}u$ is an \mathcal{O}_{E_v} -generator of $H_{1,v}(\hat{M}_{E_v,\varphi}, A_v)$ and let ω_ψ° be an $L[y_i - \psi(y_i)]$ -generator of $H^\psi(\hat{M}_{E_v,\varphi}, L[y_i - \psi(y_i)])$ such that $\omega_\psi^\circ \bmod y_i - \varphi(y_i) \in H_{\text{dR}}^1(\hat{M}_{E_v,\varphi}, L)$ is an R -generator of $H_{\text{dR}}^1(\hat{M}_{E_v,\varphi}, R)$. Let $x \in L[y_i - \psi(y_i)]^\times$ such that $\omega_\psi = x \cdot \omega_\psi^\circ$. Then

$$\int_u \omega_\psi = (a \otimes 1)x \cdot \int_{u^\circ} \omega_\psi^\circ = (a \otimes 1)x \cdot \Omega(E_v, \varphi, \psi) \in \mathbb{C}_v((z - \zeta))$$

up to multiplication by an element of $R^\times + (z - \zeta) \cdot L[z - \zeta]$ by Theorem 4.13(d). The element $(a \otimes 1)x \in E_v \otimes_{Q_v} L[z - \zeta]$ lies in the ψ -component $L[y_i - \psi(y_i)]$ of the product decomposition (A.1), and in that component $a \otimes 1$ is congruent to $\psi(a)$ modulo $y_i - \psi(y_i)$. Therefore

$$v(\int_u \omega_\psi) = v(\psi(a)x \cdot \Omega(E_v, \varphi, \psi)) = Z_v(a_{E_v,\psi,\varphi}, 1) - \mu_{\text{Art},v}(a_{E_v,\psi,\varphi}) + v(\omega_\psi) + v_\psi(u).$$

by Theorem 4.20. □

To compute $v(\int_u \omega_\psi)$ for general \hat{M} we need the following

Definition 4.23. Let E_v be separable over Q_v and let $\Phi = (d_\varphi)_{\varphi \in H_{E_v}}$ be a local CM-type. For $\psi \in H_{E_v}$ let $a_{E_v,\psi,\Phi} \in \mathcal{C}(\mathcal{G}_{Q_v}, \mathbb{Q})$ and $a_{E_v,\psi,\Phi}^0 \in \mathcal{C}^0(\mathcal{G}_{Q_v}, \mathbb{Q})$ be given by the formulas

$$a_{E_v,\psi,\Phi}(g) := \sum_{\varphi \in H_{E_v}} d_\varphi \cdot a_{E_v,\psi,\varphi}(g) = d_{g\psi} \quad \text{and} \quad (4.9)$$

$$a_{E_v,\psi,\Phi}^0(g) := \frac{1}{\#H_L} \sum_{\eta \in H_L} d_{\eta^{-1}g\eta\psi}. \quad (4.10)$$

Note that $a_{E_v,\psi,\Phi}$ and $a_{E_v,\psi,\Phi}^0$ factor through $\text{Gal}(E_v^{\text{nor}}/Q_v)$ where E_v^{nor} is the Galois closure of $\psi(E_v)$ in Q_v^{alg} . In particular, $a_{E_v,\psi,\Phi}^0$ does not depend on the field L provided $\psi(E_v) \subset L$ for all $\psi \in H_{E_v}$.

For general \hat{M} we can now prove the following

Theorem 4.24. Let \hat{M} be a local shtuka over R with complex multiplication by the ring of integers \mathcal{O}_{E_v} in a commutative, semi-simple, separable Q_v -algebra E_v with CM-type Φ , and assume that $\psi(E_v) \subset L$ for all $\psi \in H_{E_v}$ and that L is separable over Q_v . Let $u \in H_{1,v}(\hat{M}, Q_v)$ be an E_v -generator and let ω_ψ be an $L[y_{i(\psi)} - \psi(y_{i(\psi)})]$ -generator of $H^\psi(\hat{M}, L[y_{i(\psi)} - \psi(y_{i(\psi)})])$. Then the period $\int_u \omega_\psi := u \otimes \text{id}_{\mathbb{C}_v((z-\zeta))}(h_{v,\text{dR}}^{-1}(\omega_\psi))$ has valuation

$$v(\int_u \omega_\psi) = Z_v(a_{E_v,\psi,\Phi}, 1) - \mu_{\text{Art},v}(a_{E_v,\psi,\Phi}) + v(\omega_\psi) + v_\psi(u),$$

where $v(\omega_\psi)$ and $v_\psi(u)$ were defined in Definitions 3.9 and 4.21.

Proof. As in 4.1 the local shtuka \hat{M} is isomorphic to the tensor product $\hat{M}_{E_v,0} \otimes \bigotimes_{\varphi} \hat{M}_{E_v,\varphi}^{\otimes d_\varphi}$ over $\mathcal{O}_{E_v,R}$. Let $i := i(\psi)$ and $j := j(\psi)$. For every $\hat{M}_{E_v,\varphi}$ we fix the $L[y_i - \psi(y_i)]$ -generator $\omega_{\psi,\varphi}^\circ := 1 \in H^\psi(\hat{M}_{E_v,\varphi}, L[y_i - \psi(y_i)])$. In addition, we let $\omega_{\psi,0}^\circ := 1 \in H^\psi(\hat{M}_{E_v,0}, L[y_i - \psi(y_i)])$. Then we can take the tensor product $\omega_\psi^\circ := \omega_{\psi,0}^\circ \otimes \bigotimes_{\varphi \in H_{E_v}} (\omega_{\psi,\varphi}^\circ)^{\otimes d_\varphi}$ in

$$H^\psi(\hat{M}, L[y_i - \psi(y_i)]) \cong H^\psi(\hat{M}_{E_v,0}, L[y_i - \psi(y_i)]) \otimes \bigotimes_{\varphi \in H_{E_v}} H^\psi(\hat{M}_{E_v,\varphi}, L[y_i - \psi(y_i)])^{\otimes d_\varphi}.$$

It is an $L[y_i - \psi(y_i)]$ -generator as in Definition 3.9. Let $x \in L[y_i - \psi(y_i)]^\times$ be such that $\omega_\psi = x \cdot \omega_\psi^\circ$, and let further $a \in E_v$ be such that $u^\circ := a^{-1}u$ is an \mathcal{O}_{E_v} -generator of $H_{1,v}(\hat{M}, A_v)$. Then (4.5) and (4.7) imply that

$$\int_{u^\circ} \omega_\psi^\circ = \varepsilon_{i,j} c_{i,j}^{-1} \prod_{\varphi \in H_{E_v,i}} \Omega(E_v, \varphi, \psi)^{d_\varphi}$$

up to multiplication by an element of $R^\times + (z - \zeta) \cdot L[[z - \zeta]]$. Since $\varepsilon_{i,j} c_{i,j}^{-1} \in (\mathcal{O}_{E_v} \otimes_{A_v} \mathcal{O}_{\mathbb{C}_v}[[z]])^\times$ we conclude as in the proof of Corollary 4.22 that $\int_u \omega_\psi = (a \otimes 1)x \cdot \int_{u^\circ} \omega_\psi^\circ$ and

$$\begin{aligned} v(\int_u \omega_\psi) &= v\left(\psi(a)x \cdot \varepsilon_{i,j} c_{i,j}^{-1} \prod_{\varphi \in H_{E_v,i}} \Omega(E_v, \varphi, \psi)^{d_\varphi}\right) \\ &= v(\omega_\psi) + v_\psi(u) + \sum_{\varphi \in H_{E_v,i}} (Z_v(a_{E_v,\psi,\varphi}, 1) - \mu_{\text{Art},v}(a_{E_v,\psi,\varphi})) \cdot d_\varphi \\ &= Z_v(a_{E_v,\psi,\Phi}, 1) - \mu_{\text{Art},v}(a_{E_v,\psi,\Phi}) + v(\omega_\psi) + v_\psi(u), \end{aligned}$$

because $i(\varphi) \neq i(\psi)$ implies that $a_{E_v,\psi,\varphi}(g) = \delta_{g\psi,\varphi} = 0$ for all g . \square

Corollary 4.25. *Keep the situation of Theorem 4.24. For every $\eta \in H_L$ note that $i(\eta\psi) = i(\psi)$, let \hat{M}^η and $\omega_\psi^\eta \in H^{\eta\psi}(\hat{M}^\eta, L[[y_{i(\psi)} - \eta\psi(y_{i(\psi)})]])$ be obtained by extension of scalars via η , and choose an E_v -generator $u_\eta \in H_{1,v}(\hat{M}^\eta, Q_v)$. Then*

$$\frac{1}{\#H_L} \sum_{\eta \in H_L} v(\int_{u_\eta} \omega_\psi^\eta) = Z_v(a_{E_v,\psi,\Phi}^0, 1) - \mu_{\text{Art},v}(a_{E_v,\psi,\Phi}^0) + \frac{1}{\#H_L} \sum_{\eta \in H_L} (v(\omega_\psi^\eta) + v_{\eta\psi}(u_\eta)).$$

Proof. Since \hat{M}^η has complex multiplication by \mathcal{O}_{E_v} with CM-type $\eta\Phi := (d'_\varphi)_{\varphi \in H_{E_v}}$ with $d'_\varphi = d_{\eta^{-1}\varphi}$, Theorem 4.24 implies

$$v(\int_{u_\eta} \omega_\psi^\eta) = Z_v(a_{E_v,\eta\psi,\eta\Phi}, 1) - \mu_{\text{Art},v}(a_{E_v,\eta\psi,\eta\Phi}) + v(\omega_\psi^\eta) + v_{\eta\psi}(u_\eta).$$

Summing over all η and observing that $a_{E_v,\eta\psi,\eta\Phi}(g) = d'_{g\eta\psi} = d_{\eta^{-1}g\eta\psi}$ proves the corollary. \square

Finally we are ready to give the

Proof of Theorem 1.3. This follows by applying Corollary 4.25 to $\hat{M} = \hat{M}_v(\mathcal{M})$ for a model \mathcal{M} of \underline{M} with good reduction, to $\omega_\psi \otimes_K L \in H^\psi(\underline{M}, L[[y_\psi - \psi(y_\psi)]] = H^\psi(\hat{M}, L[[y_{i(\psi)} - \psi(y_{i(\psi)})]])$, to $h_{\text{Betti},v}(u_\eta) \in H_{1,v}(\hat{M}^\eta, Q_v)$, and to $E_v := E \otimes_Q Q_v$. \square

A Appendix: Product Decompositions of Certain Rings

In this appendix we establish certain product decompositions for the rings used in this article. We begin with the following

Lemma A.1. *Let k be a field and let $z = \sum_{n=0}^\infty b_n y^n \in k[[y]]$. Let $\psi: k[[y]] \rightarrow R$ be a ring homomorphism into a k -algebra R . Then in $k[[y]] \hat{\otimes}_{k,\psi} R := \varprojlim_n k[[y]]/(y^n) \otimes_{k,\psi} R \cong R[[y]]$ the fraction $\frac{z \otimes 1 - 1 \otimes \psi(z)}{y \otimes 1 - 1 \otimes \psi(y)}$ exists and is congruent to $1 \otimes \psi\left(\frac{dz}{dy}\right)$ modulo $y \otimes 1 - 1 \otimes \psi(y)$.*

Proof. The lemma follows from the computation

$$\begin{aligned} z \otimes 1 - 1 \otimes \psi(z) &= \sum_{n=0}^\infty (b_n y^n \otimes 1 - 1 \otimes \psi(b_n) \psi(y)^n) \\ &= \sum_{n=1}^\infty (1 \otimes \psi(b_n)) \cdot \sum_{\nu=0}^{n-1} (y^\nu \otimes \psi(y)^{n-1-\nu}) \cdot (y \otimes 1 - 1 \otimes \psi(y)) \\ &= (y \otimes 1 - 1 \otimes \psi(y)) \cdot \sum_{\nu=0}^\infty y^\nu \otimes \psi\left(\sum_{n=\nu+1}^\infty b_n y^{n-1-\nu}\right), \end{aligned}$$

where the second factor converges in $k[[y]] \widehat{\otimes}_{k,\psi} R$. Modulo $y \otimes 1 - 1 \otimes \psi(y)$ this factor equals

$$\begin{aligned} \sum_{n=1}^{\infty} (1 \otimes \psi(b_n)) \cdot \sum_{\nu=0}^{n-1} (y^\nu \otimes \psi(y)^{n-1-\nu}) &= \sum_{n=1}^{\infty} (1 \otimes \psi(b_n)) \cdot n (1 \otimes \psi(y)^{n-1}) \\ &= 1 \otimes \psi \left(\sum_{n=1}^{\infty} n b_n y^{n-1} \right) \\ &= 1 \otimes \psi \left(\frac{dz}{dy} \right). \end{aligned}$$

□

We need the following well known fact from field theory. For the convenience of the reader we include a proof.

Lemma A.2. *Let E be a finite field extension of Q (or of Q_v) of inseparability degree p^m . Then the separable closure E' of Q (resp. of Q_v) in E equals $E^{p^m} := \{x^{p^m} : x \in E\}$. If y is a uniformizing parameter at a place \tilde{v} of E then $y' := y^{p^m}$ is a uniformizing parameter at the place \tilde{v}' of E' below \tilde{v} and $E = E'(y) = E'[X]/(X^{p^m} - y')$.*

Proof. This is due to the fact that Q has transcendence degree one over \mathbb{F}_q , respectively that Q_v is a discretely valued field. Namely, consider the case for Q_v . Then $E = k((y))$ where k is the finite residue field of E . Clearly $E^{p^m} = k((y'))$ and $E = E^{p^m}(y) = E^{p^m}[X]/(X^{p^m} - y')$, because $X^{p^m} - y'$ is irreducible in $E^{p^m}[X]$ by Eisenstein. In particular $[E : E^{p^m}] = p^m$. On the other hand, the minimal polynomial $f(X)$ of y over E' is of the form $g(X^{p^{m'}})$ for a separable, irreducible polynomial g over E' . Therefore the minimal polynomial of $y^{p^{m'}}$ over E' is g and $y^{p^{m'}}$ is separable over E' . This implies $y^{p^{m'}} \in E'$ and $\deg g = 1$, whence $\deg f = p^{m'} \leq p^m$. Therefore $m' \leq m$ and $y^{p^m} \in E'$, and hence $E^{p^m} \subset E'$. Since $[E : E'] = p^m = [E : E^{p^m}]$ it follows that $E' = E^{p^m}$. This proves the lemma for Q_v .

For Q a proof for the equality $E' = E^{p^m}$ can be found for example in [Sil86, Chapter II, Corollary 2.12]. If $\mathcal{O}_{E,\tilde{v}}$ is the valuation ring of E at \tilde{v} then $(\mathcal{O}_{E,\tilde{v}})^{p^m}$ equals the valuation ring $\mathcal{O}_{E',\tilde{v}'}$ of E' at \tilde{v}' and so y' is a uniformizing parameter of $\mathcal{O}_{E',\tilde{v}'}$. The last equality follows from the fact that the polynomial $X^{p^m} - y' \in \mathcal{O}_{E',\tilde{v}'}[X]$ is irreducible by Eisenstein. □

In the next lemma we consider the embeddings $Q \hookrightarrow K[[z_v - \zeta_v]]$ and $Q_v \hookrightarrow L[[z_v - \zeta_v]]$ given by $z_v \mapsto z_v = \zeta_v + (z_v - \zeta_v)$.

Lemma A.3. *Let $E = E_1 \times \dots \times E_s$ be a product of finite field extensions of Q and let $K \subset Q^{\text{alg}}$ be a field extension of Q with $\psi(E) \subset K$ for all $\psi \in H_E := \text{Hom}_Q(E, Q^{\text{alg}})$. Let $i(\psi)$ be such that ψ factors through $E \rightarrow E_{i(\psi)}$ and let $y_{i(\psi)} \in E_{i(\psi)}$ be a uniformizing parameter at a place of $E_{i(\psi)}$ above v . Then*

$$\begin{aligned} E \otimes_Q K[[z_v - \zeta_v]] &= \prod_{\psi \in H_E} K[[y_{i(\psi)} - \psi(y_{i(\psi)})]] \quad \text{and} \\ E \otimes_Q K &= \prod_{\psi \in H_E} K[[y_{i(\psi)} - \psi(y_{i(\psi)})]] / (y_{i(\psi)} - \psi(y_{i(\psi)}))^{[E_{i(\psi)} : Q]_{\text{insep}}}, \end{aligned}$$

where $[E_{i(\psi)} : Q]_{\text{insep}}$ is the inseparability degree of $E_{i(\psi)}$ over Q .

Likewise, let $E_v = E_{v,1} \times \dots \times E_{v,s}$ be a product of finite field extensions of Q_v and let $L \subset Q_v^{\text{alg}}$ be a field extension of Q_v with $\psi(E_v) \subset L$ for all $\psi \in H_{E_v} := \text{Hom}_{Q_v}(E_v, Q_v^{\text{alg}})$. Let $i(\psi)$ be such that ψ factors through $E \rightarrow E_{v,i(\psi)}$ and let $y_{i(\psi)} \in E_{v,i(\psi)}$ be a uniformizing parameter. Then

$$E_v \otimes_{Q_v} L[[z_v - \zeta_v]] = \prod_{\psi \in H_{E_v}} L[[y_{i(\psi)} - \psi(y_{i(\psi)})]] \quad \text{and} \tag{A.1}$$

$$E_v \otimes_{Q_v} L = \prod_{\psi \in H_{E_v}} L[[y_{i(\psi)} - \psi(y_{i(\psi)})]] / (y_{i(\psi)} - \psi(y_{i(\psi)}))^{[E_{v,i(\psi)} : Q_v]_{\text{insep}}}, \tag{A.2}$$

where $[E_{v,i(\psi)} : Q_v]_{\text{insep}}$ is the inseparability degree of $E_{v,i(\psi)}$ over Q_v .

Proof. Fix a ψ , set $i := i(\psi)$ and let E'_i , respectively $E'_{v,i}$ be the separable closure of Q in E_i , respectively of Q_v in $E_{v,i}$. Then $H_{E_i} = H_{E'_i}$, respectively $H_{E_{v,i}} = H_{E'_{v,i}}$, and

$$E'_i \otimes_Q K \xrightarrow{\sim} \prod_{\psi \in H_{E_i}} K, \quad \text{respectively} \quad E'_{v,i} \otimes_{Q_v} L \xrightarrow{\sim} \prod_{\psi \in H_{E_{v,i}}} L. \quad (\text{A.3})$$

Let $p^m := [E_i : Q]_{\text{insep}} = [E_i : E'_i]$, respectively $p^m := [E_{v,i} : Q_v]_{\text{insep}} = [E_{v,i} : E'_{v,i}]$, and let $y'_i := y_i^{p^m}$. Then Lemma A.2 implies that $y'_i \in E'_i$ is a uniformizing parameter at a place above v . By Hensel's lemma the decompositions (A.3) extend to decompositions

$$\begin{aligned} E'_i \otimes_Q K[[z_v - \zeta_v]] &\xrightarrow{\sim} \prod_{\psi \in H_{E_i}} K[[z_v - \zeta_v]] = \prod_{\psi \in H_{E_i}} K[[y'_i - \psi(y'_i)]], \quad \text{respectively} \\ E'_{v,i} \otimes_{Q_v} L[[z_v - \zeta_v]] &\xrightarrow{\sim} \prod_{\psi \in H_{E_{v,i}}} L[[z_v - \zeta_v]] = \prod_{\psi \in H_{E_{v,i}}} L[[y'_i - \psi(y'_i)]]. \end{aligned}$$

Here the last identifications in each line follow from Lemma A.1 by observing that the derivative $\frac{dz}{dy'_i}$ equals $-\frac{\partial}{\partial y'_i} m(z, y'_i) / \frac{\partial}{\partial z} m(z, y'_i)$ where $m(z, y'_i)$ is the minimal polynomial of y'_i over Q , respectively over Q_v , and hence $\psi(\frac{dz}{dy'_i})$ is non-zero by the separability of y'_i over Q , respectively over Q_v , and the injectivity of ψ on E_i , respectively on $E_{v,i}$. Now $E_i = E'_i(y_i)$, respectively $E_{v,i} = E'_{v,i}(y_i)$, and hence $E_i \otimes_{E'_i} K[[y'_i - \psi(y'_i)]] = K[[y'_i - \psi(y'_i)]] [y_i - \psi(y_i)] = K[[y_i - \psi(y_i)]]$, respectively $E_{v,i} \otimes_{E'_{v,i}} L[[y'_i - \psi(y'_i)]] = L[[y'_i - \psi(y'_i)]] [y_i - \psi(y_i)] = L[[y_i - \psi(y_i)]]$ with $(y_i - \psi(y_i))^{p^m} = y'_i - \psi(y'_i)$. Since $H_E = \bigcup_i H_{E_i}$, respectively $H_{E_v} = \bigcup_i H_{E_{v,i}}$, the lemma follows. \square

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